

Mathematics

Lecture 6

MAT 112

Series Convergence Test and Taylor Series

Prepared by
Dr. Israa Lewaa

Lecture Objectives

- 1- The first aim is to know well what we mean by infinite series
 - 2- Secondly, The definition of convergent and divergent
 - 3- It is important to know some notes when solving limits
 - 4- Know the different Convergent tests
 - 5- Taylor series
-

Lecture Outline

1- Infinite Series

2- Convergent and Divergent Definition

3- Different Convergent tests

4- Taylor series

Infinite Series

An infinite series is the sum of an infinite sequence of numbers:

$$a_1 + a_2 + \dots + a_n + \dots$$

How can we find this sum of infinite numbers in a finite life time? For his, we look at the sequence of sums of finite number of terms, called the sequence of partial sums:

$$\sum_{n=1}^{\infty} a_n$$

Illustrative Example

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$$

The Sum for the first two items is 3

The Sum for the first three items is 6

The Sum for the first 100 items is 5050

The Sum for the first 1000 items is 500500

It is clear that by increasing the number of items, the sum will tend to infinity

So, this series is divergent.

Illustrative Example

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

The Sum for the first two items is 1.5

The Sum for the first three items is 1.75

The Sum for the first 10 items is 1.99805

The Sum for the first 30 items is 1.9999

It is clear that by increasing the number of items, the sum will not be greater than 2

So, this series is Convergent at 2.

Strategy for Testing Series

1- nth Term Test

2- Direct Comparison Test

3- Limit Comparison Test

4- Cauchy Test

5- Integral Test

6- Ratio Test

7- Root Test

8- Alternating Series Test

Strategy for Testing Series

we will learn about:

The ways of testing a series for convergence or divergence.

Nth term Test

Limit Comparison Test

Cauchy Test

We will need some important notes for limits

Important Note for Limits

- $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n^2} = e$
- $\lim_{n \rightarrow \infty} \frac{7n+8}{8n+13} = \frac{7}{8}$
- $\lim_{n \rightarrow \infty} \frac{n^3+n^2}{2n^3+3n-4} = \frac{1}{2}$

Important Note for Limits

- $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{3n-1}} = \frac{1}{\sqrt{3}}$
- $\lim_{n \rightarrow \infty} \left(\frac{2n}{4n-1}\right)^{\frac{2n-1}{n}} = \left(\frac{2}{4}\right)^2 = \frac{1}{4}$
- $\lim_{n \rightarrow \infty} \left(\frac{5n}{7n-1}\right)^{\frac{3n-1}{n}} = \left(\frac{5}{7}\right)^3 = \frac{125}{343}$

Note for Some Series

Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

Example A

Test Convergent and Divergent of the below P series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Answer: Since $p > 1$, So it is converges

Example B

Test Convergent and Divergent of the below P series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Answer: Since $p < 1$, So it is diverges

1- nth Term Test

Limit of n th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Limit of n th-Term Test for Divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 1

Test Convergent and Divergent of the below series using nth term Test

$$\sum_{n=1}^{\infty} \frac{2n + 3}{5n + 11}$$

Example 1 Solution

$$\sum_{n=1}^{\infty} \frac{2n+3}{5n+11} = \frac{5}{16} + \frac{7}{21} + \frac{9}{26} + \dots$$

$$\lim_{n \rightarrow \infty} \frac{2n+3}{5n+11} = \frac{2}{5} \neq 0$$

So, it is Divergence

2- Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = L$$

where L is *finite and positive*. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 2

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3 + 3n - 4}$$

Example 2 Solution

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3 + 3n - 4} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{n^2}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{n+1}{2n^3 + 3n - 4} \times \frac{n^2}{1} = \frac{n^3 + n^2}{2n^3 + 3n - 4} = \frac{1}{2} \neq 0$$

Example 2 Solution

The limit is not equal to zero and positive

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3+3n-4} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ are both convergent or divergent}$$

*Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **is** convergent (see Example A)*

Then $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+3n-4}$ is convergent

3- Cauchy Test

$$\lim_{n \rightarrow \infty} a^n = L$$

$L < 1$ *it will be convergent*

$L > 1$ *it will be divergent*

$L = 1$ *Test failed*

Example 3

Test Convergent and Divergent of the below series using Cauchy test

$$\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$$

Example 3 Solution

$$\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$$

By using Cauchy test

$$\lim_{n \rightarrow \infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}} = \left(\frac{2}{4} \right)^2 = \frac{1}{4} < 1$$

Then $\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$ is convergent

Extra Practice – Example 4

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Example 4 Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{n^2}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{1}{n(n+3)} \times \frac{n^2}{1} = \frac{n^2}{n^2 + 3n} = 1 \neq 0$$

Example 4 Solution

The limit is not equal to zero and positive

$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are both convergent or divergent

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (see Example A)

Then $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ is convergent

Extra Practice – Example 5

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$$

Example 5 Solution

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{\sqrt{n}}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{3n-1}} \times \frac{\sqrt{n}}{1} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{3n-1}} = \frac{1}{\sqrt{3}} \neq 0$$

Example 5 Solution

The limit is not equal to zero and positive

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ are both convergent or divergent}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (see Example B)

Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$ is divergent

Extra Practice – Example 6

Test Convergent and Divergent of the below series using Cauchy test

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

Example 6 Solution

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} = 2 + \frac{3^4}{2} + \frac{4^9}{3} + \dots$$

By using Cauchy test

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n^2} = e > 1$$

Then $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$ is divergent

Taylor Series

Definitions of n th Taylor Polynomial

If f has n derivatives at c , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **n th Taylor polynomial for f at c .**

Example 1 – Taylor Series

Let $f(x) = x^3$

$C = 1$

Find $P_2(x)$ as $n = 2$

Example 1 Solution

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2$$

as $n = 2$ and $C = 1$

$$P_2(x) = 1 + 3(x - 1) + \frac{6}{2!}(x - 1)^2$$

$$P_2(x) = 1 + 3(x - 1) + \frac{6}{2}(x^2 - 2x + 1)$$

$$P_2(x) = 1 + 3x - 3 + 3(x^2 - 2x + 1)$$

$$P_2(x) = 1 + 3x - 3 + 3x^2 - 6x + 3$$

$$***$P_2(x) = 3x^2 - 3x + 1$***$$

$F(x) = x^3$	$F(c) = 1^3 = 1$
$f'(x) = 3x^2$	$f'(c) = 3(1)^2 = 3$
$f''(x) = 6x$	$f''(c) = 6(1) = 6$

Example 2 – Taylor Series

Let $f(x) = 2x^2$

$C = 1$

Find $P_2(x)$ as $n = 2$

Example 2 Solution

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2$$

as $n = 2$ and $C = 1$

$$P_2(x) = 2 + 4(x - 1) + \frac{4}{2!}(x - 1)^2$$

$$P_2(x) = 2 + 4(x - 1) + \frac{4}{2}(x^2 - 2x + 1)$$

$$P_2(x) = 2 + 4x - 4 + 2(x^2 - 2x + 1)$$

$$P_2(x) = 2 + 4x - 4 + 2x^2 - 4x + 2$$

$$**$P_2(x) = 2x^2$**$$

$F(x) = 2x^2$	$F(c) = 2 * 1^2 = 2$
$f'(x) = 4x$	$f'(c) = 4 * 1 = 4$
$f''(x) = 4$	$f''(c) = 4$

Lecture Summary

1- Infinite Series

2- Convergent and Divergent Definition

Lecture Summary

3- Different Convergent tests

Nth term Test

Limit Comparison Test

Cauchy Test

Lecture Summary

3- Different Convergent tests

Nth term Test

Limit Comparison Test

Cauchy Test

4- Taylor series

Important Announcement

Differential Revision is essential