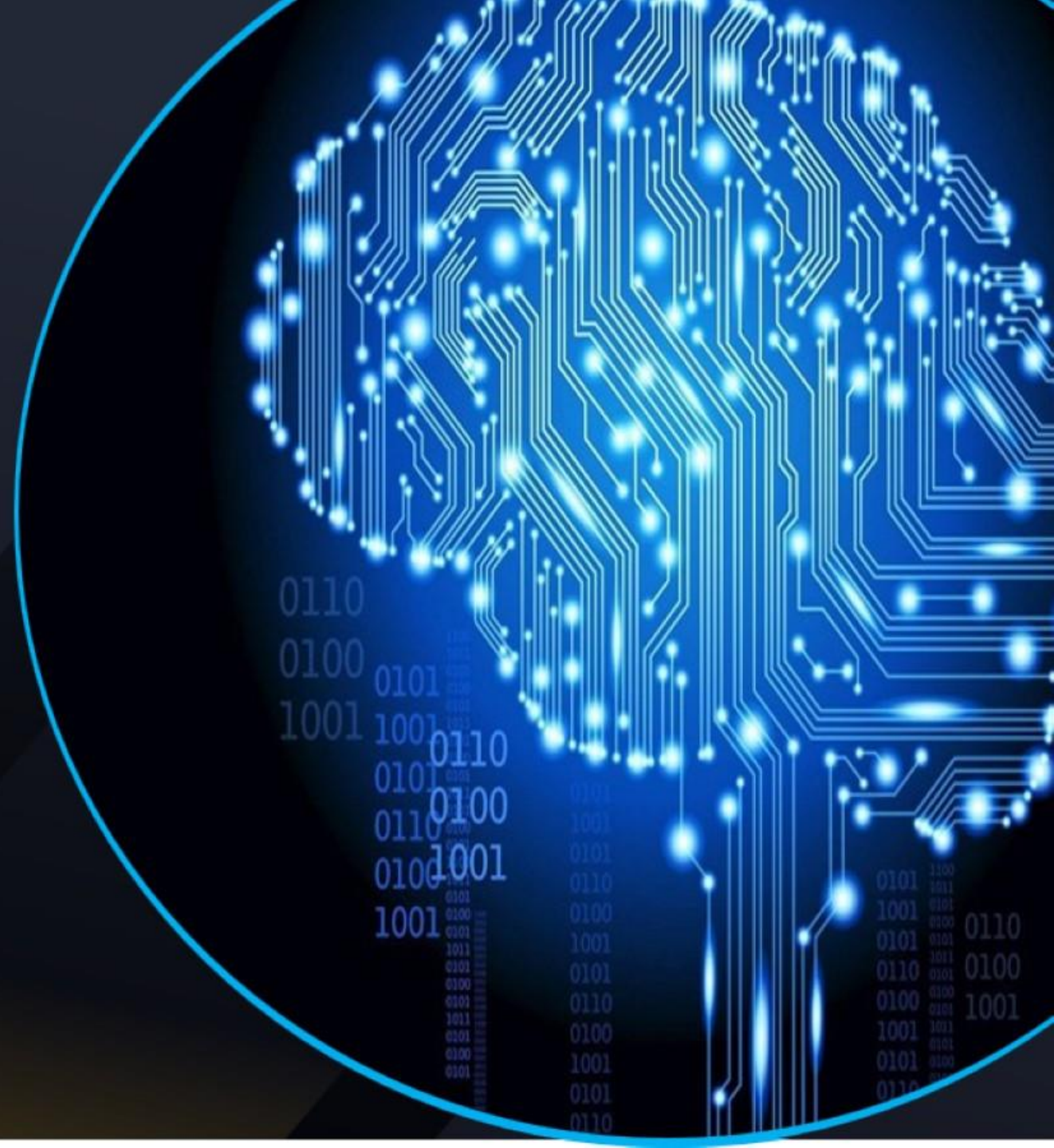


Discrete Mathematics

Dr. Shahira Ahmed

IT241

Lecture 3





Discrete Mathematics

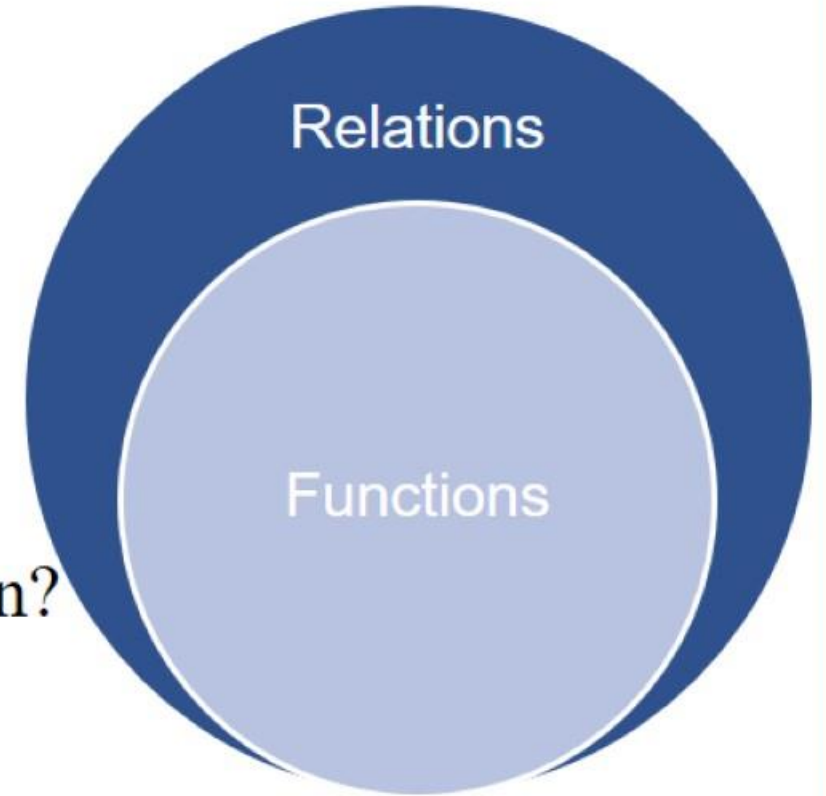
IT241

Lecture 3: Functions

Dr. Shahira Ahmed

- Definition of a Function.
 - Domain, Codomain
 - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial

- A function is a relation where exactly one element of B is related to each element of A .

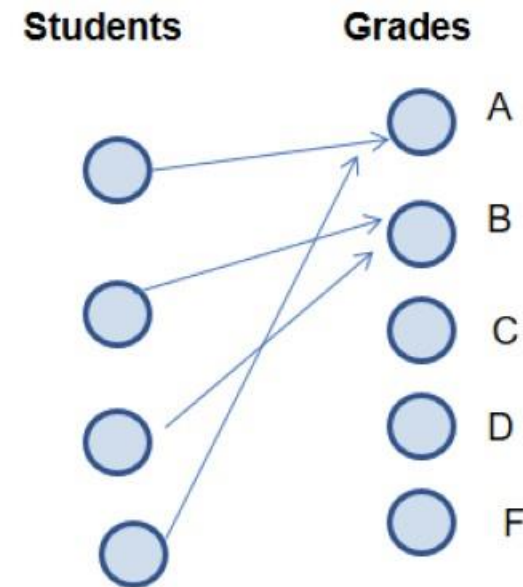


- Example $R = \{(2, 1), (3, 6), (2, 3), (5, 8)\}$ is a function?
- Solution False

Definition: Let A and B be nonempty sets. A *function* f from A to B , denoted $f: A \rightarrow B$ is an assignment of each element of A to exactly one element of B .

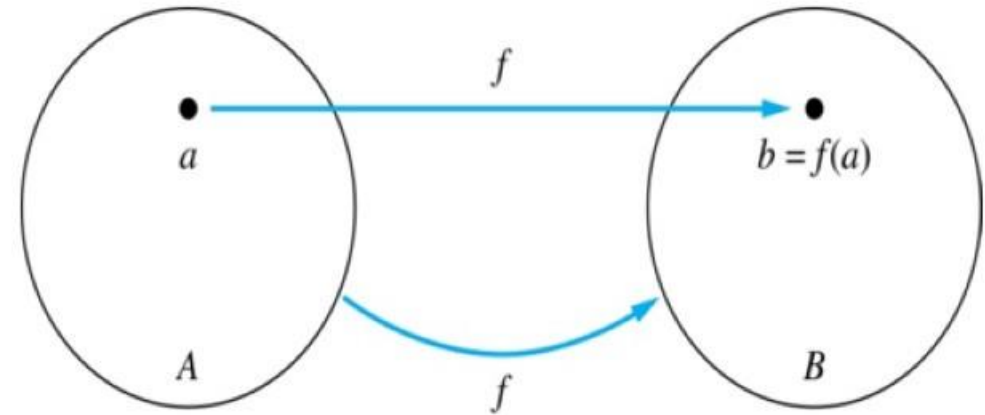
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A .

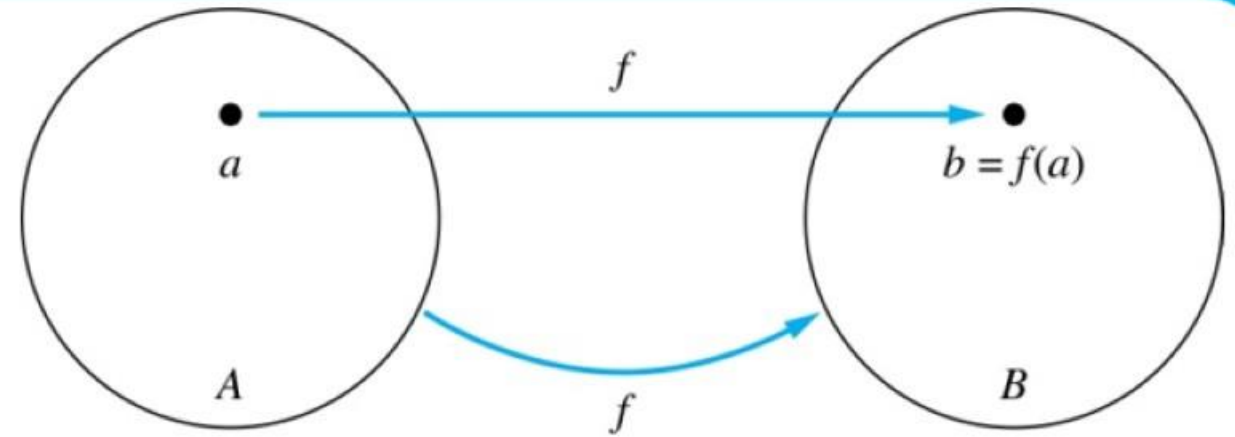
- Functions are sometimes called *mappings* or *transformations*.



Given a function $f: A \rightarrow B$:

- We say f maps A to B or
 f is a *mapping* from A to B .
- A is called the **domain** of f .
- B is called the **codomain** of f .

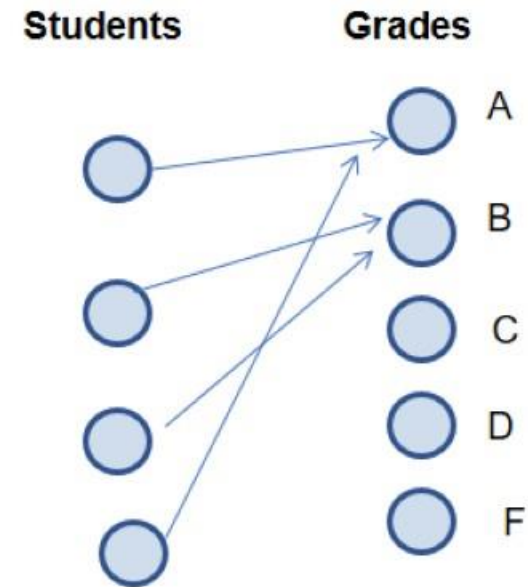




- If $f(a) = b$,
 - then b is called the **image** of a under f .
 - a is called the **preimage** of b .
- The **range** of f is the set of all images of points in A under f . We denote it by $f(A)$.

Codomain vs Range

- Codomain= $\{A,B,C,D,F\}$
- Range= $\{A,B\}$



- Functions may be specified in different ways:

- An explicit statement of the assignment.

Students and grades example.

- A formula.

$$f(x) = x + 1$$

- A computer program.

- A Java program that when given an integer n , produces the n th Fibonacci Number

$f(a) = ?$ z

The image of d is ? z

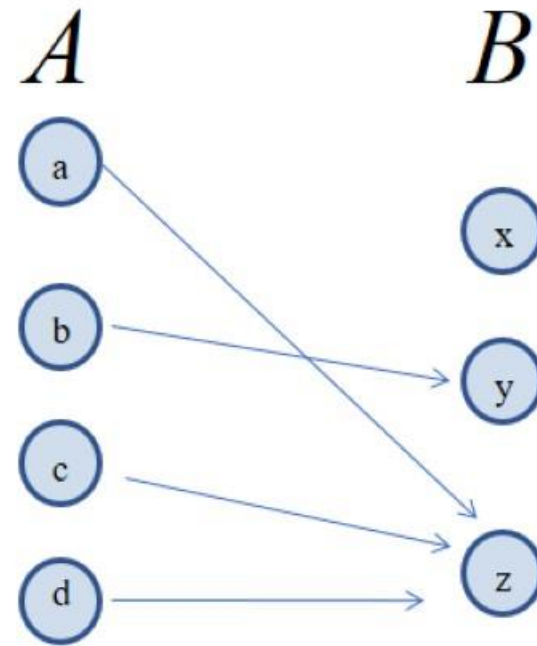
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

Range $\rightarrow f(A) = ?$ $\{y, z\}$

The preimage(s) of z is (are) ? $\{a, c, d\}$

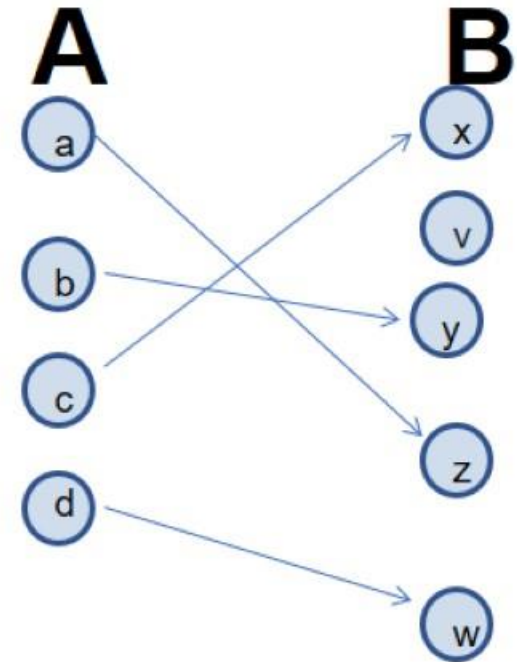


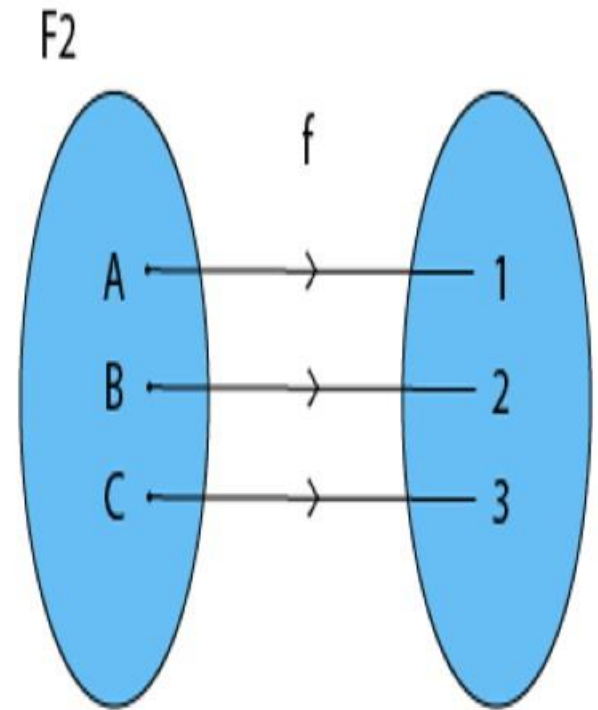
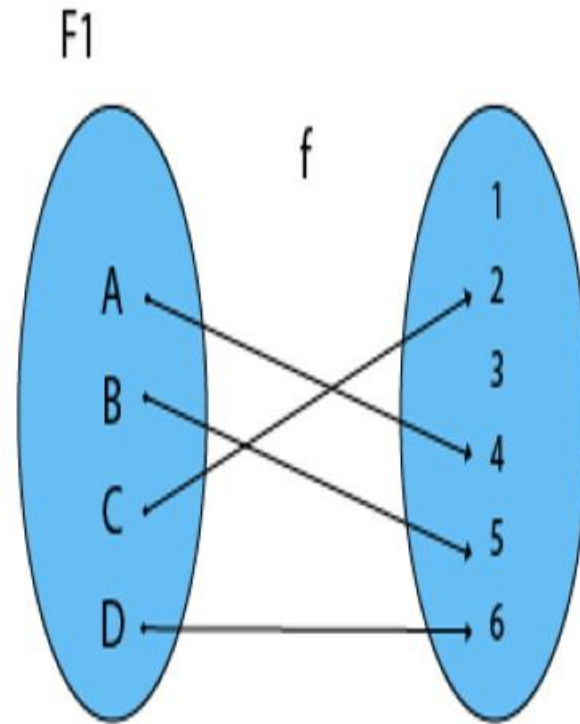
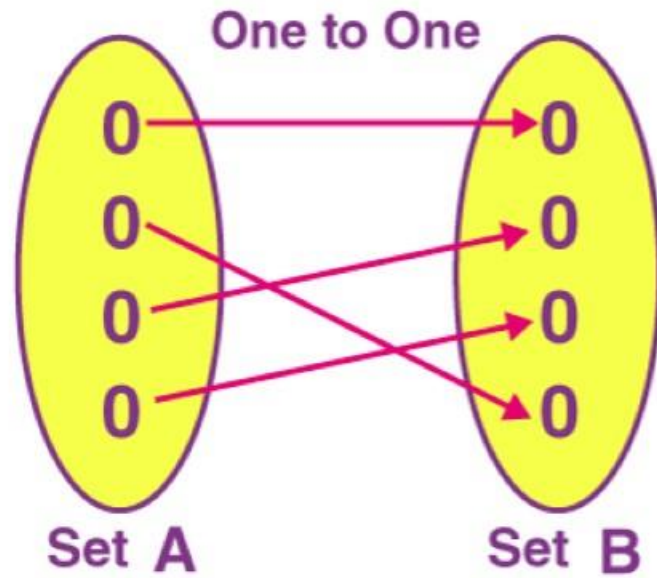
- Let R be the relation with ordered pairs
(A, 22), (B, 24), (C, 21), (D, 22), (E, 24), and (F, 22).
Here each pair consists of a graduate student and this student's age.
- Write function of this relation.
Solution:
 - $f(A) = 22$
 - $f(B) = 24$
 - $f(C) = 21$
 - $f(D) = 22,$
 - $f(E) = 24$
 - $f(F) = 22.$
- [Here, $f(x)$ is the age of x , where x is a student.]

- Let R be the relation with ordered pairs $(A, 22)$, $(B, 24)$, $(C, 21)$, $(D, 22)$, $(E, 24)$, and $(F, 22)$. Here each pair consists of a graduate student and this student's age.
- Find Domain, codomain, Range
- Solution
 - Domain = $\{A, B, C, D, E, F\}$.
 - Codomain
 - the set of positive integers between 20-90
 - The range = $\{21, 22, 24\}$

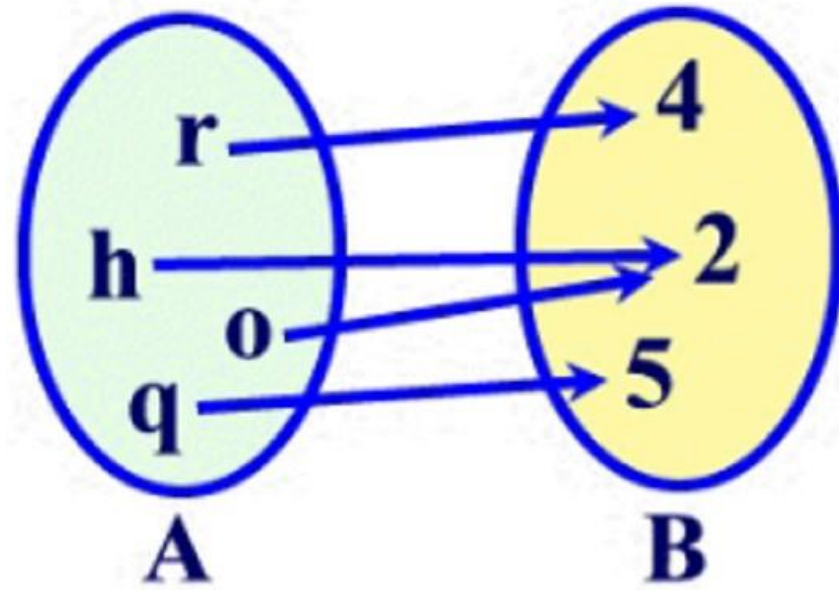
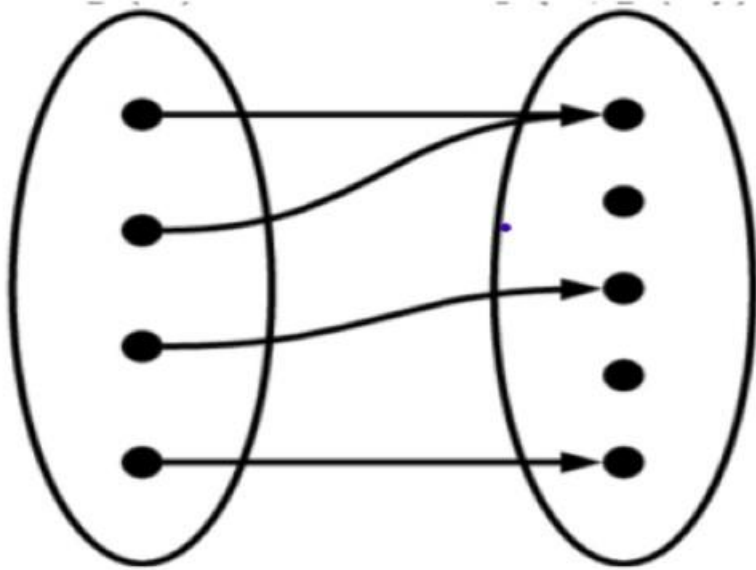
One-to-One Functions (Injections)

Definition: A function f is said to be *one-to-one*, or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f . A function is said to be an *injection* if it is one-to-one.



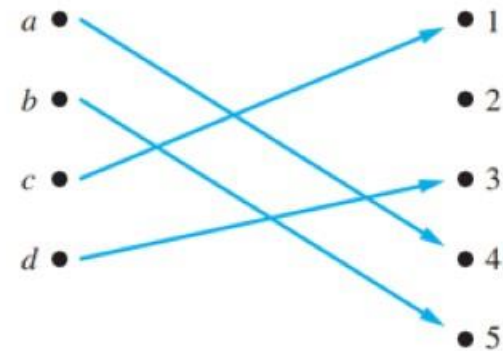


Injection



Not Injection

- Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.
- Solution
 - True

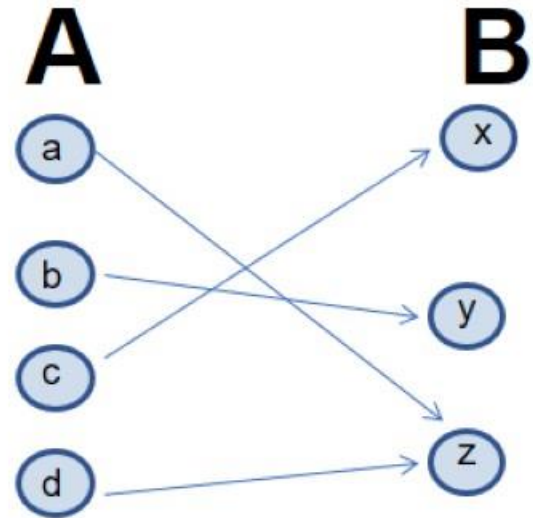


- Determine whether the function $f(x) = x^2$ $f: \mathbb{Z} \rightarrow \mathbb{Z}$ from the set of integers to the set of integers is one-to-one.
- Solution
 - False $\rightarrow f(1) = f(-1) = 1$, but $1 \neq -1$

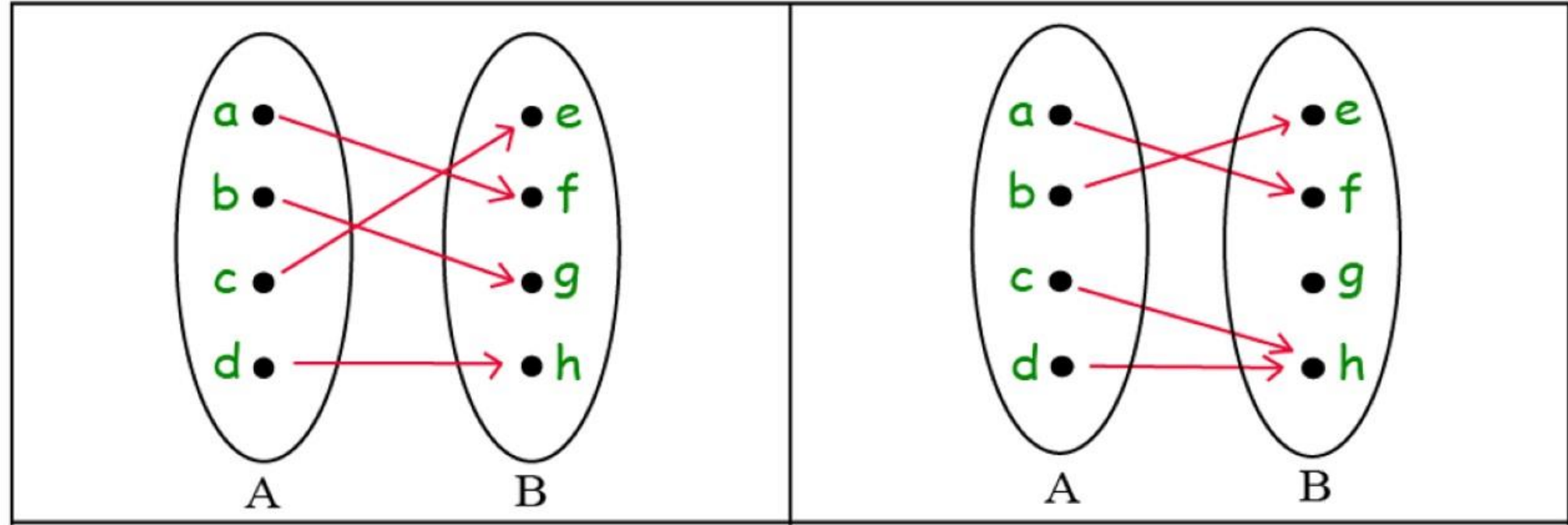
- Determine whether the function $f(x) = x^2$. $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}$ from the set of positive integers to the set of integers is one-to-one?
- Solution
 - True

- Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.
- Solution
 - True

Definition: A function f from A to B is called *onto* or *surjective*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.



- **Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 1$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?**
- **Solution**
 - **True**



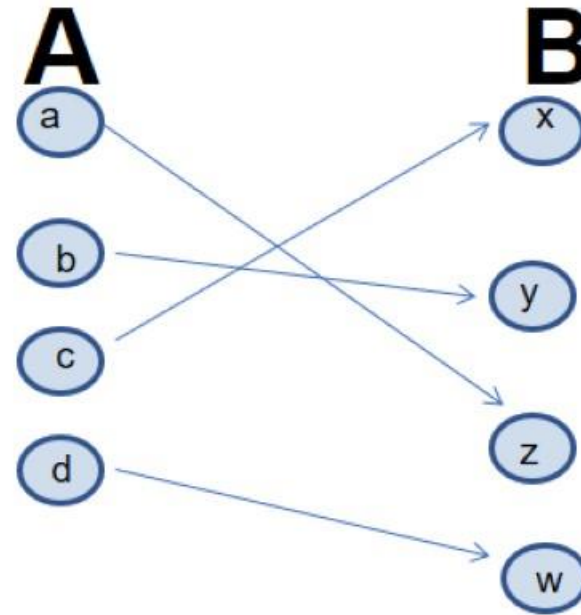
surjective

Not surjective

- Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is onto.
- Solution
 - True

- Determine whether the function $f(x) = x^2$ from the set of integers to itself is onto.
- Solution
 - False
 - f is not onto because there is no integer x with $x^2 = -1$, for example.

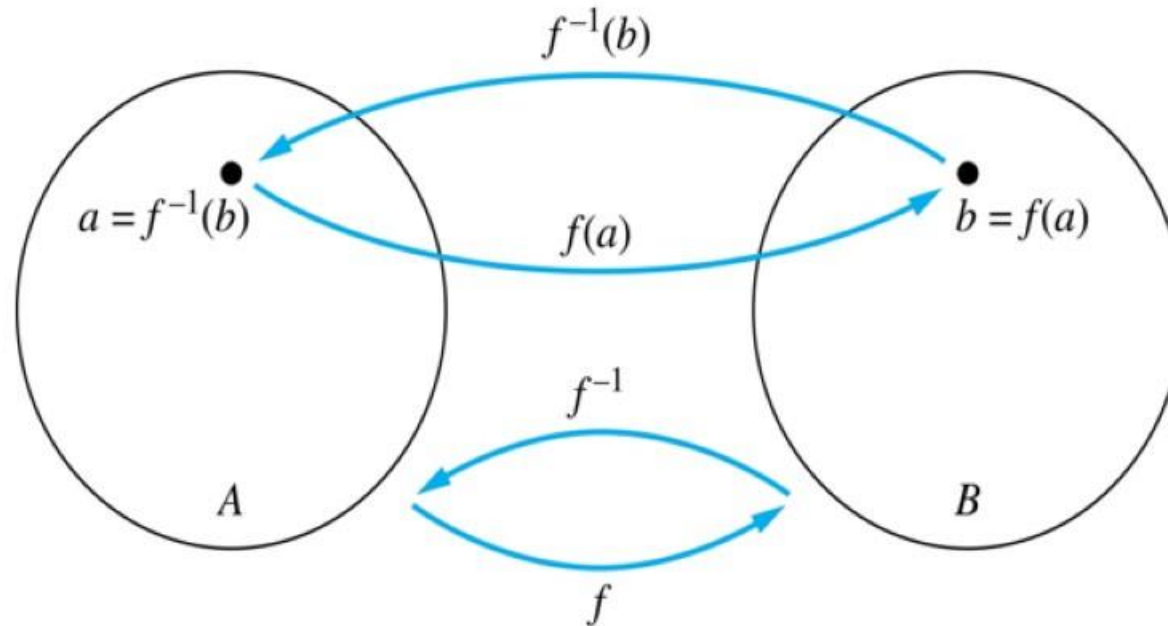
Definition: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).



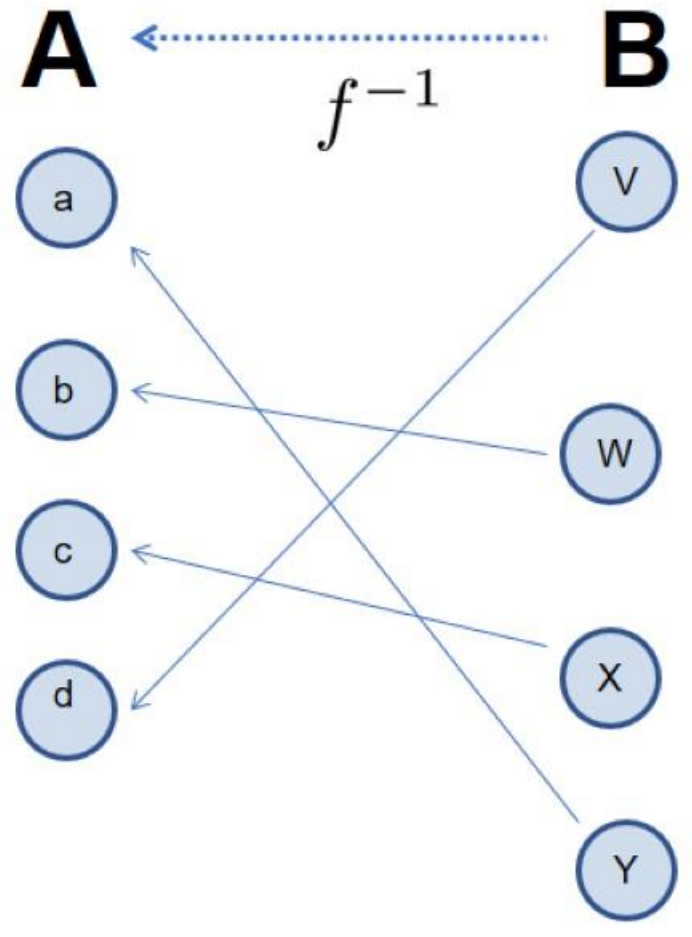
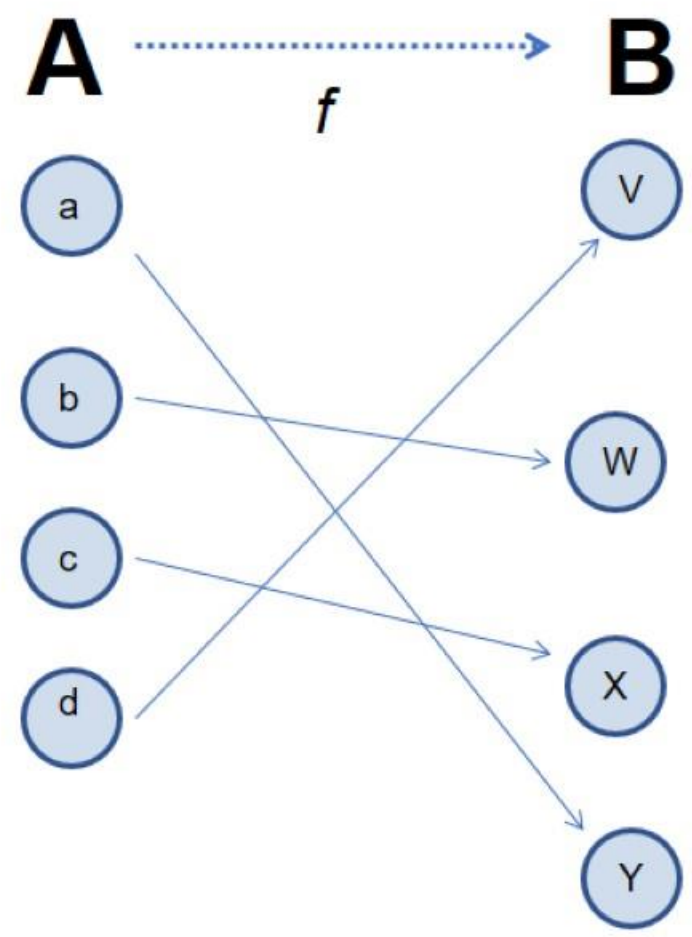
- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?
- Solution:
 - The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value.
 - It is onto because all four elements of the codomain are images of elements in the domain.
 - Hence, f is a bijection.

Definition: Let f be a bijection from A to B . Then the *inverse* of f , denoted f^{-1} , is the function from B to A defined as $f^{-1}(y) = x$ iff $f(x) = y$

No inverse exists unless f is a bijection. **Why?**



Inverse Functions



Let f be the function from $\{a,b,c\}$ to $\{1,2,3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f , so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

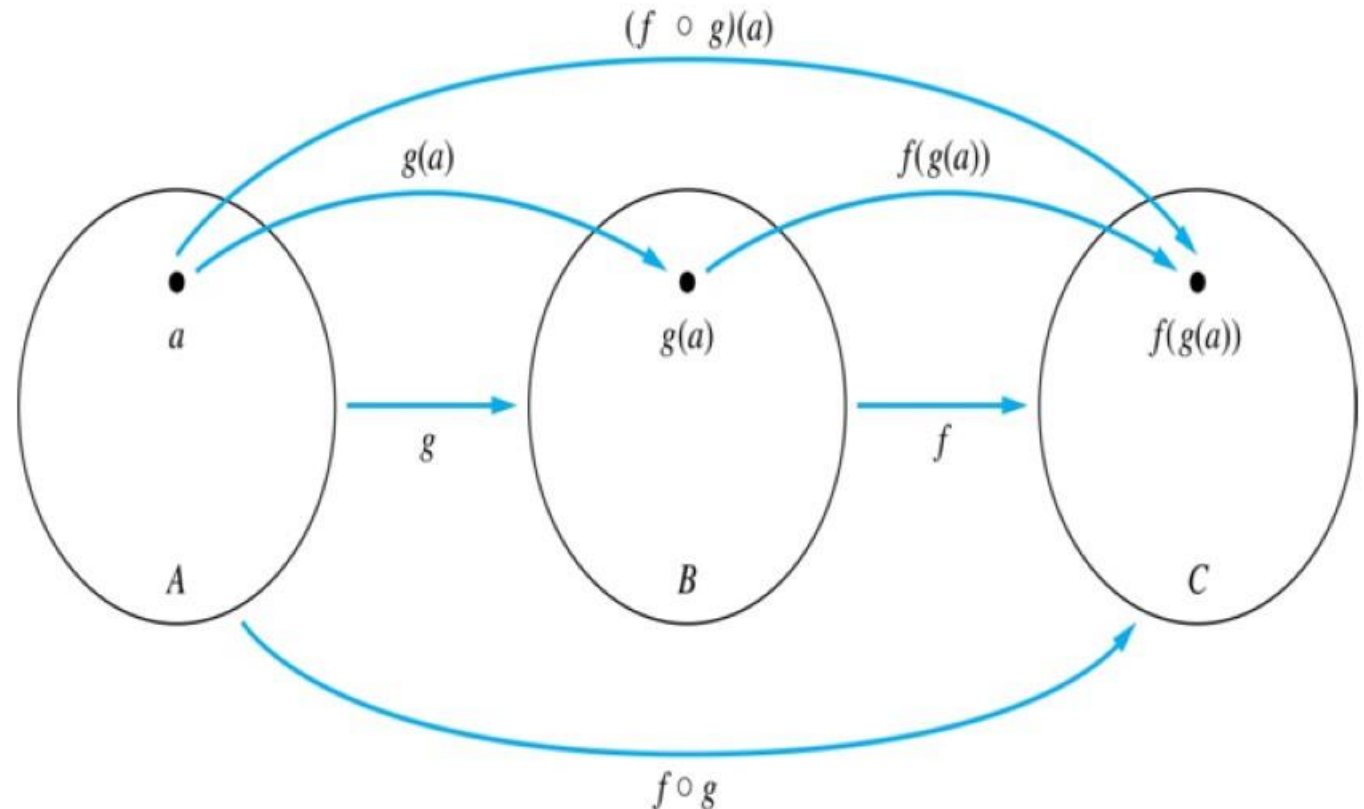
Example 3: Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: False, The function f is not invertible because it is not one-to-one.

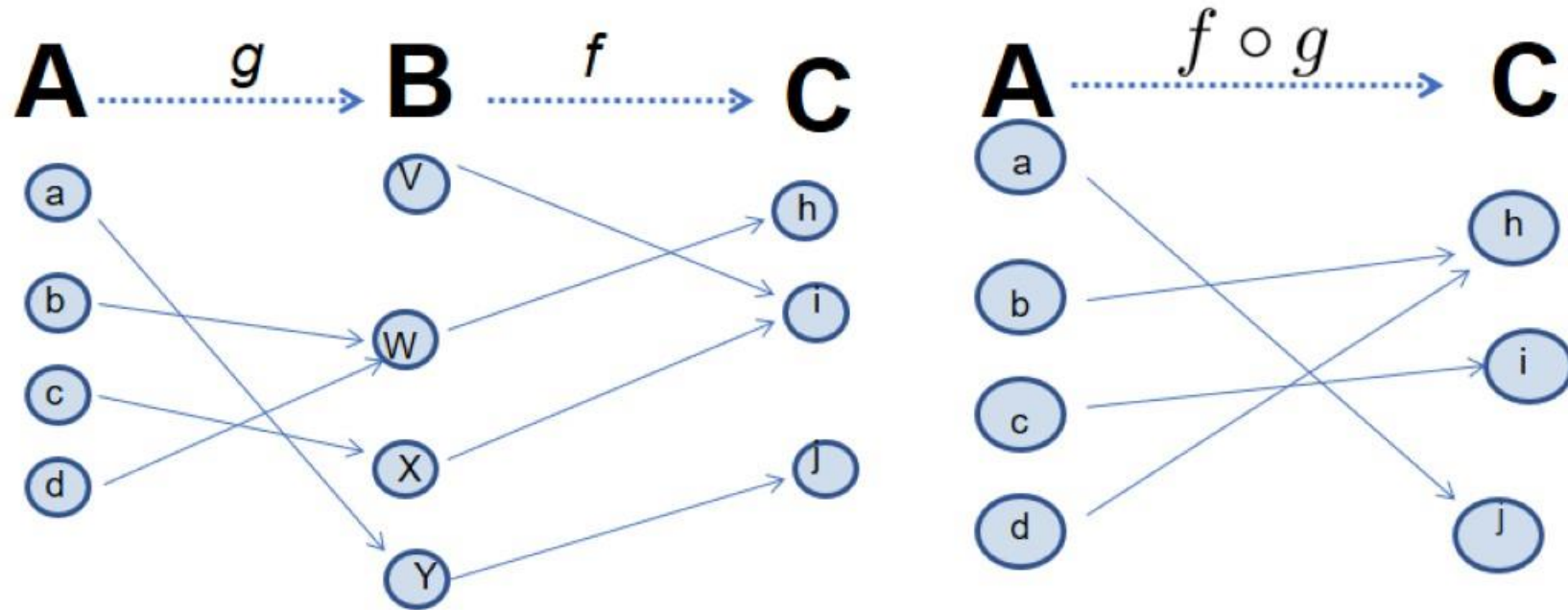
- Definition:** Let $f: B \rightarrow C$, $g: A \rightarrow B$. The *composition of f with g* , denoted is the function from A to C defined by

$$f \circ g$$

$$f \circ g(x) = f(g(x))$$



Composition



Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g , and what is the composition of g and f .

Solution: The composition $f \circ g$ is defined by

$$f \circ g (a) = f(g(a)) = f(b) = 2.$$

$$f \circ g (b) = f(g(b)) = f(c) = 1.$$

$$f \circ g (c) = f(g(c)) = f(a) = 3.$$

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g .

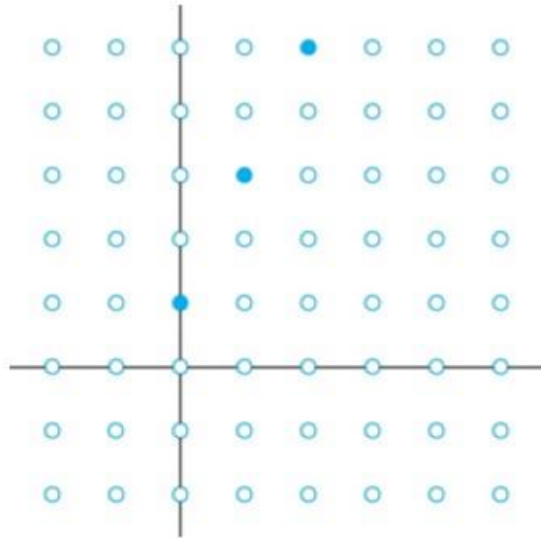
If $f(x) = x^2$ and $g(x) = 2x + 1$, then Find

- $f \circ g$
- $g \circ f$

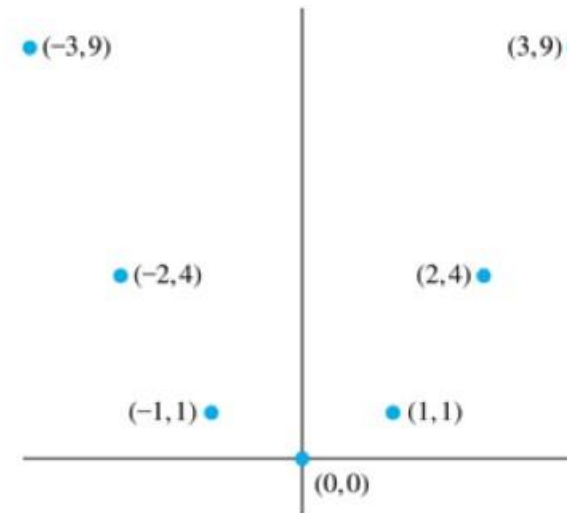
$$f(g(x)) = (2x + 1)^2$$

$$g(f(x)) = 2x^2 + 1$$

- Let f be a function from the set A to the set B . The *graph* of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of $f(n) = 2n + 1$
from $\mathbb{Z}$ to $\mathbb{Z}$



Graph of $f(x) = x^2$
from $\mathbb{Z}$ to $\mathbb{Z}$

Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g , and also the composition of g and f ?

Solution:

$$f \circ g(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$.

What is the composition of f and g , and also the composition of g and f ?

Solution:

$$f \circ g(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

$$g \circ f(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Some Important Functions

- The *floor* function, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to x .
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x

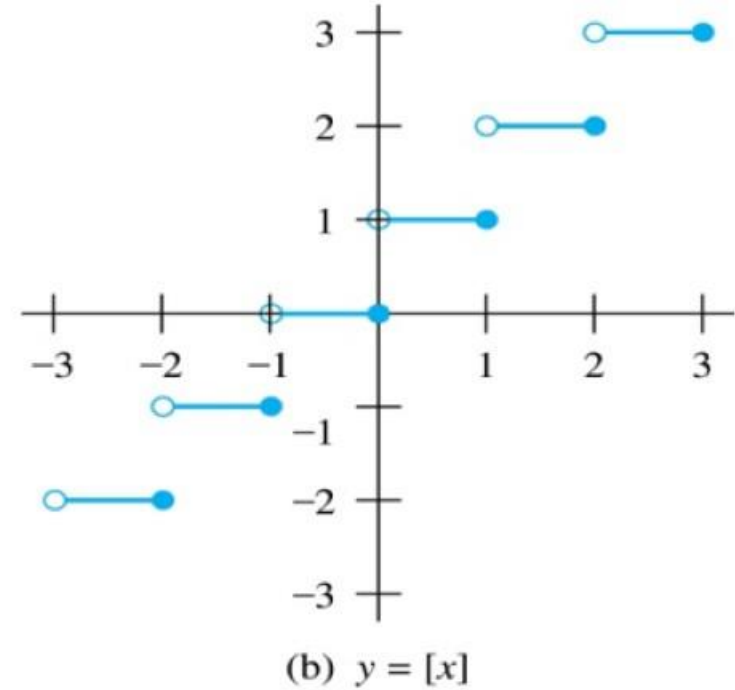
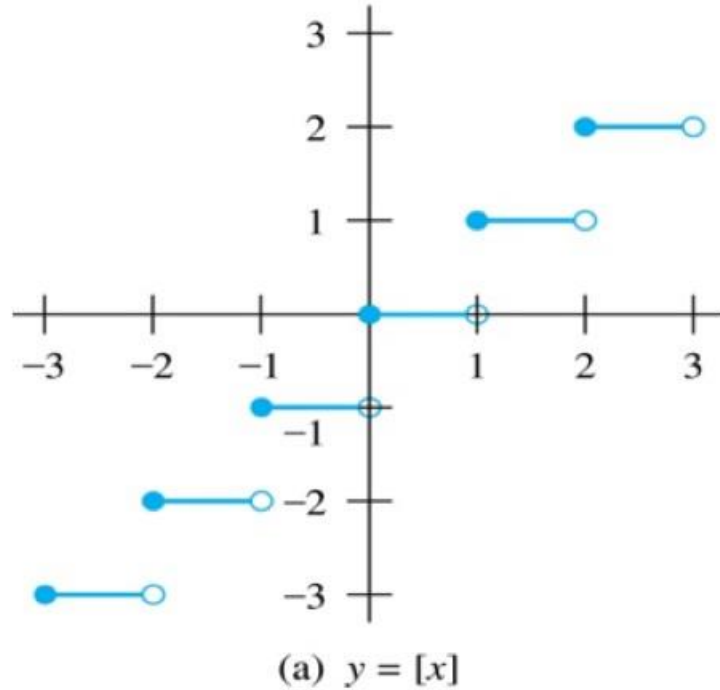
Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$
 $\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Some Important Functions

- The *floor* function, denoted $f(x) = \lfloor x \rfloor$ is the largest integer less than or equal to x .
- The *ceiling* function, denoted $f(x) = \lceil x \rceil$ is the smallest integer greater than or equal to x

Example: $\lceil 3.5 \rceil = 4$ $\lfloor 3.5 \rfloor = 3$
 $\lceil -1.5 \rceil = -1$ $\lfloor -1.5 \rfloor = -2$

Floor and Ceiling Functions



Graph of (a) Floor and (b) Ceiling Functions

Definition: $f: \mathbf{N} \rightarrow \mathbf{Z}^+$, denoted by $f(n) = n!$ is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \quad f(0) = 0! = 1$$

Examples:

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2,432,902,008,176,640,000.$$

- Find
 - $\lfloor \frac{1}{2} \rfloor, \lceil \frac{1}{2} \rceil,$
 - $\lfloor -\frac{1}{2} \rfloor, \lceil -\frac{1}{2} \rceil,$
 - $\lfloor 3.1 \rfloor, \lceil 3.1 \rceil,$
 - $\lfloor 7 \rfloor, \lceil 7 \rceil$
 - Factorial(3)

- Solution

$$\lfloor \frac{1}{2} \rfloor = 0, \lceil \frac{1}{2} \rceil = 1, \lfloor -\frac{1}{2} \rfloor = -1, \lceil -\frac{1}{2} \rceil = 0, \lfloor 3.1 \rfloor = 3, \lceil 3.1 \rceil = 4, \lfloor 7 \rfloor = 7, \lceil 7 \rceil = 7.$$

- $\text{Fact}(3)=1*2*3=6$

- Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?
- Solution: To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required.

- In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?
- Solution: In 1 minute, this connection can transmit $500,000 \cdot 60 = 30,000,000$ bits. Each ATM cell is 53 bytes long, which means that it is $53 \cdot 8 = 424$ bits long. To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424. Consequently, $\lfloor 30,000,000/424 \rfloor = 70,754$ ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection.

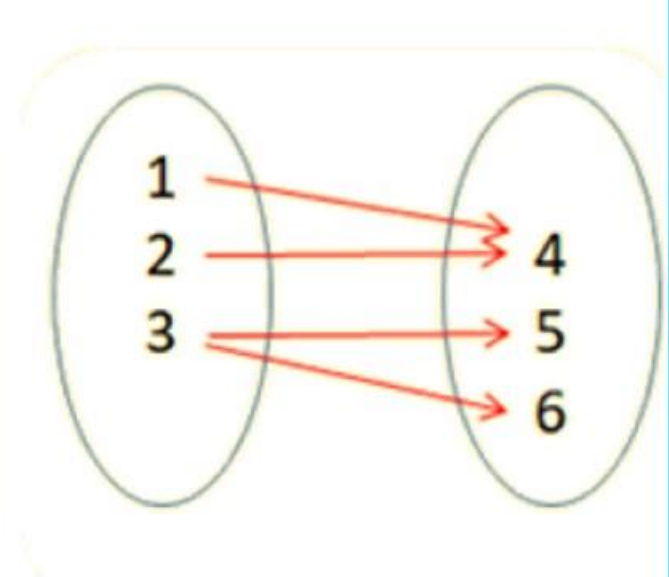
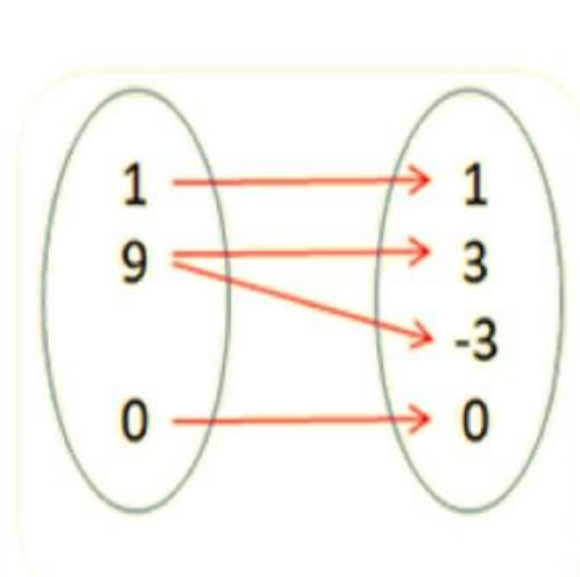
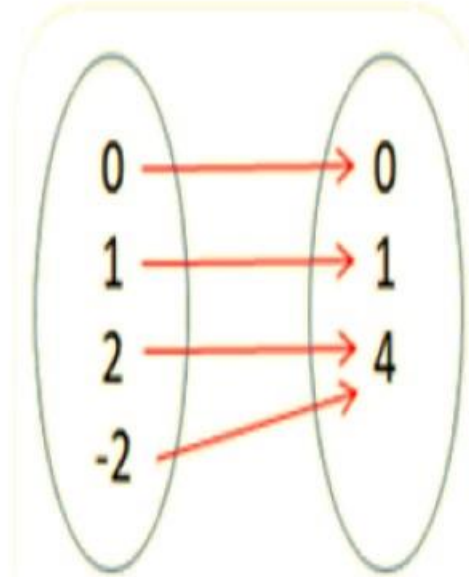
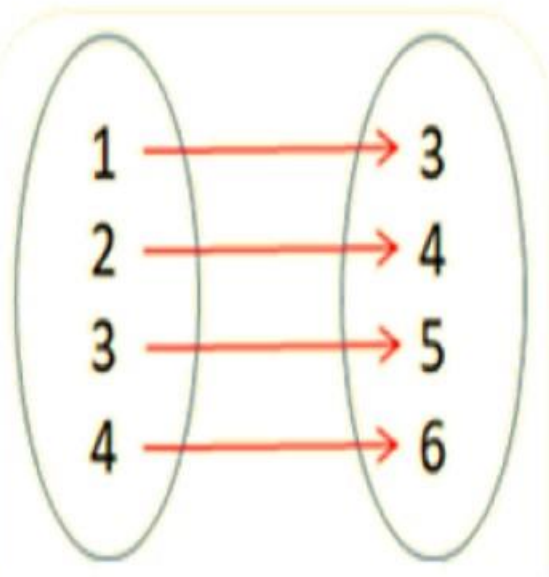
Recap

- True or False
- Given Two sets A, B
 - $A = \{v, w, z\}$
 - $B = \{1, 2, 3\}$
 - the $\{(v,1)(v,2)(v,3)\}$ is a function from $A \rightarrow B$

False

A function is a relation where exactly one element of B is related to each element of A .

Which of the following Figures Represent Function



Determine whether the following are functions

a) $A = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$

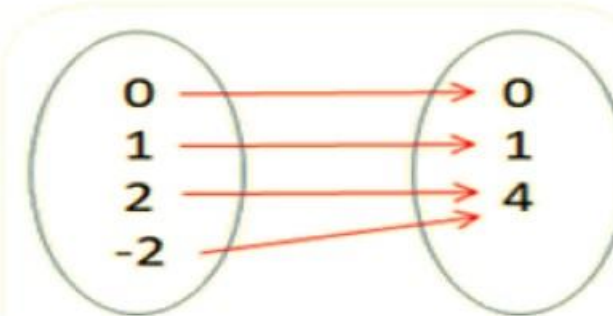
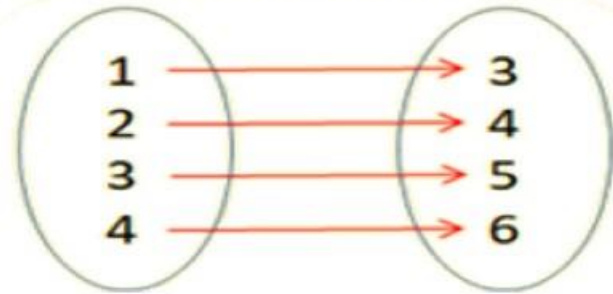
b) $B = \{(1, 3), (0, 3), (2, 1), (4, 2)\}$

c) $C = \{(1, 6), (2, 5), (1, 9), (4, 3)\}$

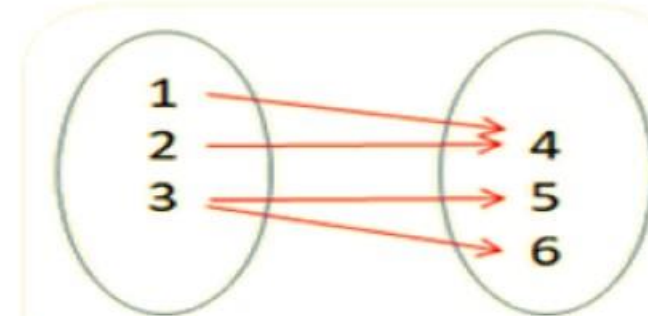
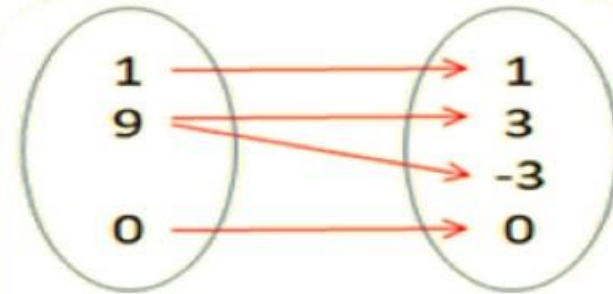
Solution:

- a) $A = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$ is a function because all the first elements are different.
- b) $B = \{(1, 3), (0, 3), (2, 1), (4, 2)\}$ is a function because all the first elements are different.
(The second element does not need to be unique)
- c) $C = \{(1, 6), (2, 5), (1, 9), (4, 3)\}$ is not a function because the first element, 1, is repeated.

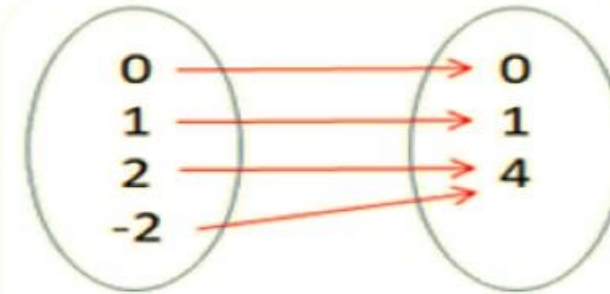
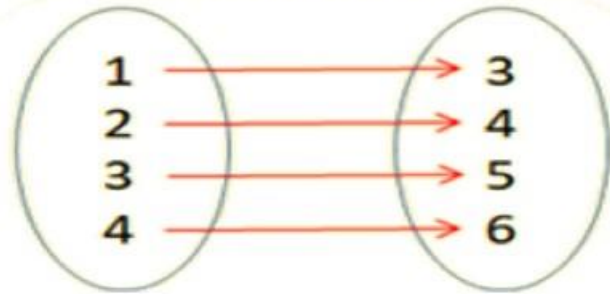
Functions



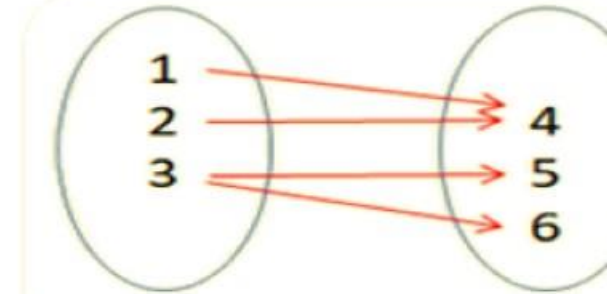
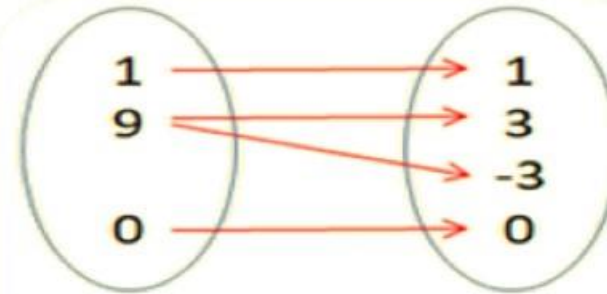
Not Functions



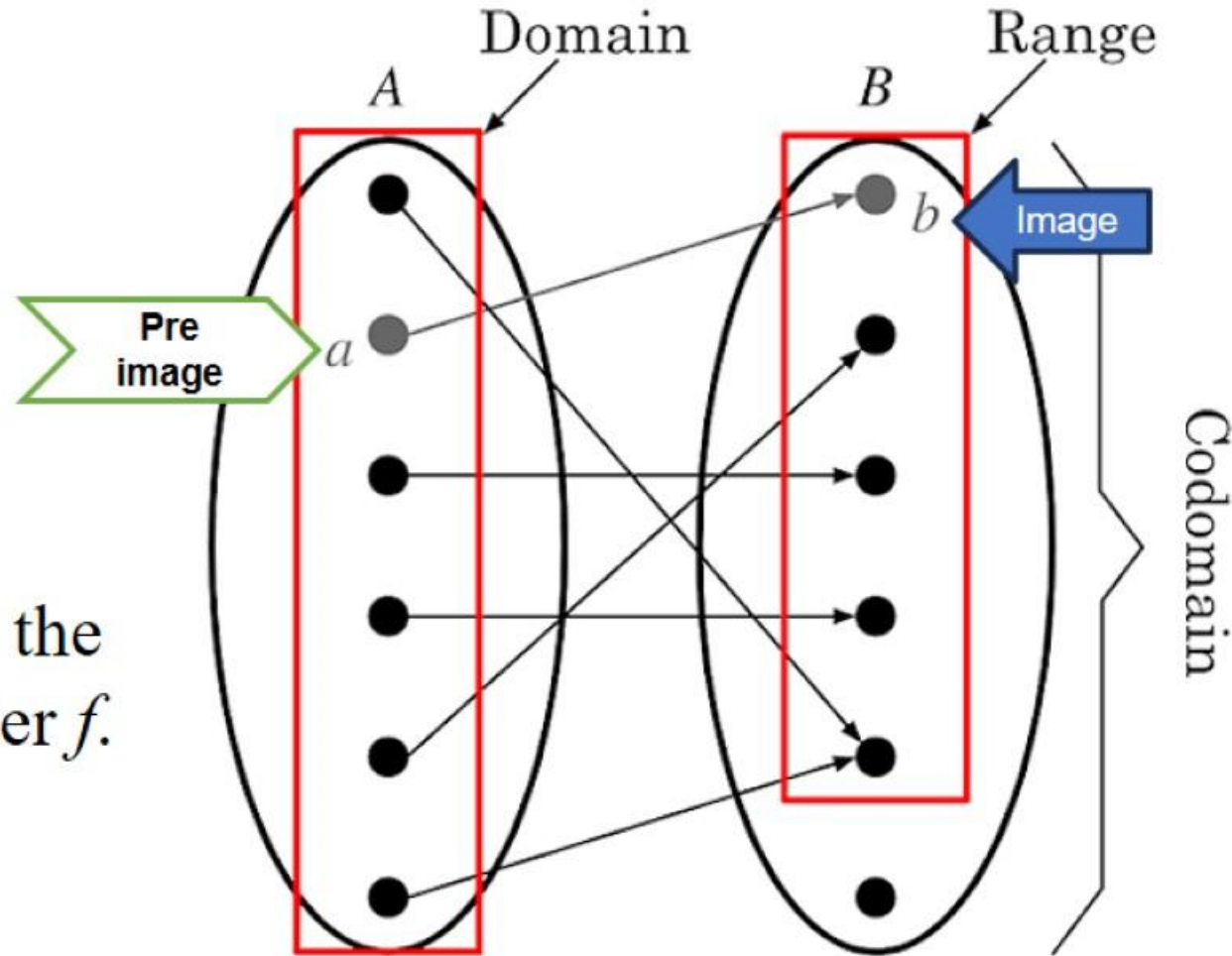
Functions



Not Functions



$$f: A \rightarrow B$$



If $f(a) = b$,
then b is called the
image of a under f .
 a is called the
preimage of b

- **Find domain and range?**

1) $f: A \rightarrow B$

- *Solution*

- *Domian :A*
- *Codomain: B*

2) $f: N \rightarrow Z^+$

- *Solution*

- *Domian :N*
- *Codomain:Z⁺*

$$f: N \rightarrow N$$

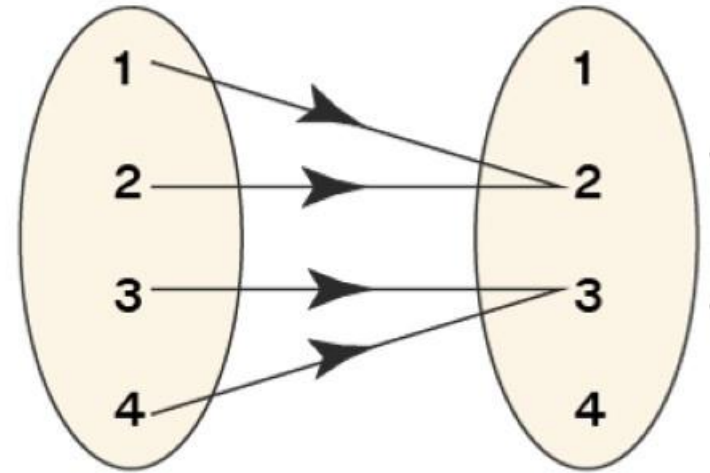
In the relation, $\{(-2, 3), (4, 5), (6, -5)\}$

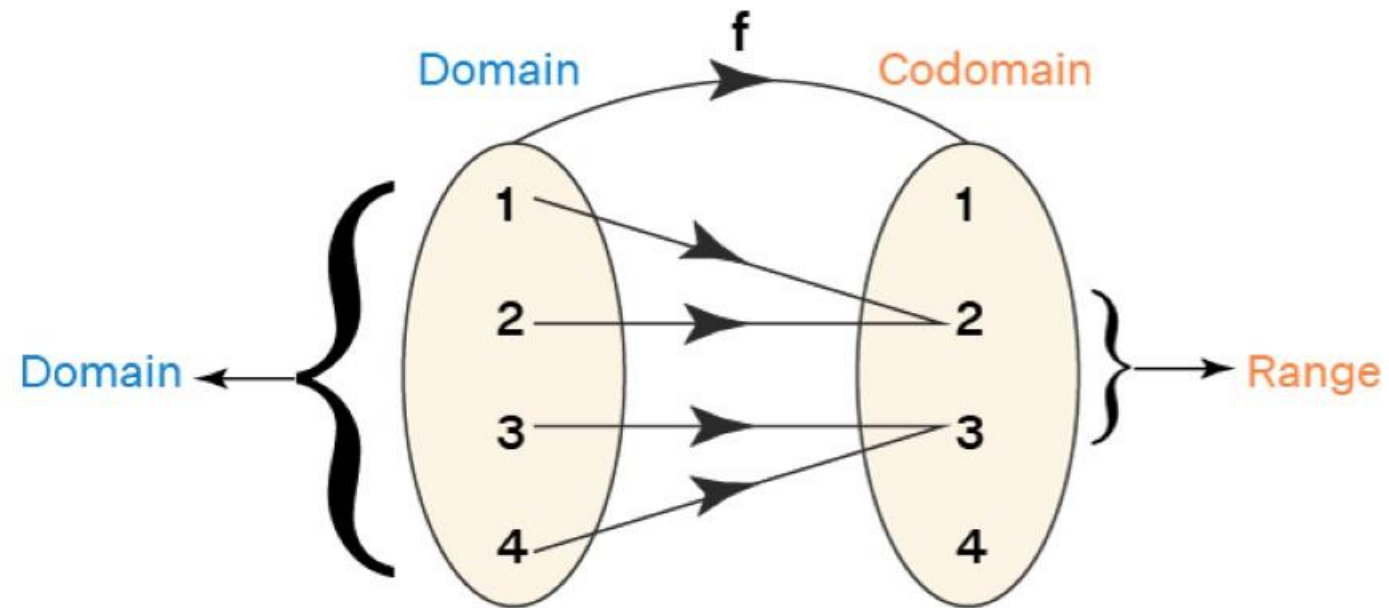
Find domain, Codomain, range, $\text{image}(-2)$, $\text{preimage}(5)$

Solution

- Domain N
- Codomain is N
- Range is $\{-5, 3, 5\}$.
- $\text{Image}(-2)$ is 3
- $\text{Preimage}(5) \rightarrow 4$

- $R = \{(1, 2), (2, 2), (3, 3), (4, 3)\}$, find domain, codomain, range, $f(1)$.
- Domain = $\{1, 2, 3, 4\}$
- Codomain = $\{1, 2, 3, 4\}$
- Range = $\{2, 3\}$
- $f(1) = \{2\}$





- Find These Values:

1. $[0.75]$

2. $[-0.75]$

3. $[3]$

4. $[0.75]$

5. $[-0.75]$

6. $[-1]$

Solution

1. 1

2. 0

3. 3

4. 0

5. -1

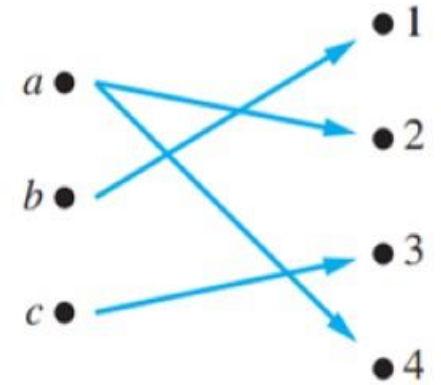
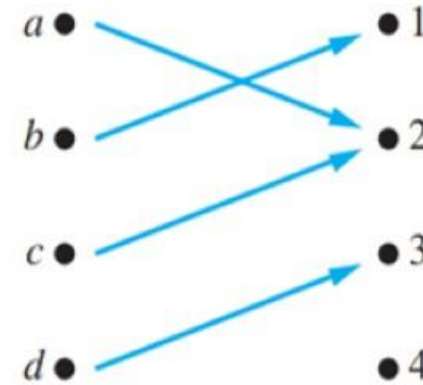
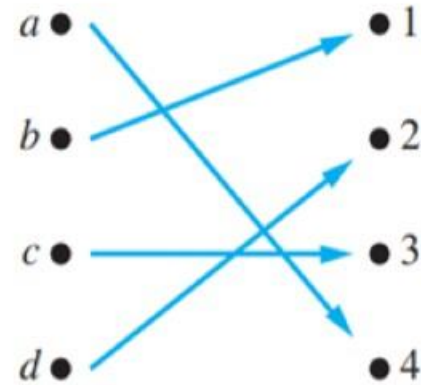
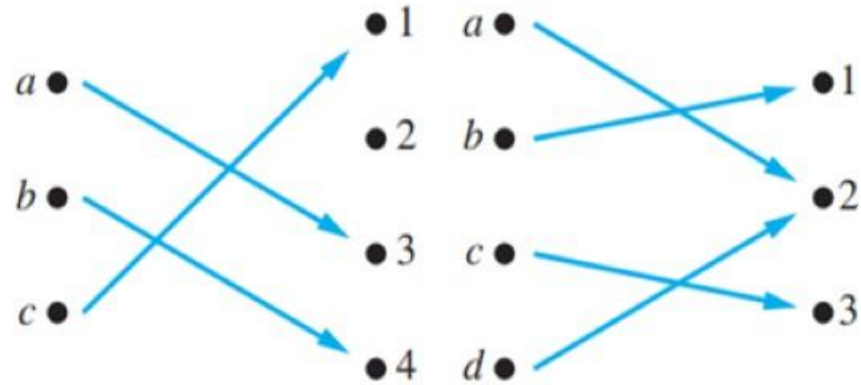
6. -1

- Determine whether each of these functions from $\{a, b, c, d\}$ to itself is onto.
 - a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$
 - b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$
 - c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

Solution

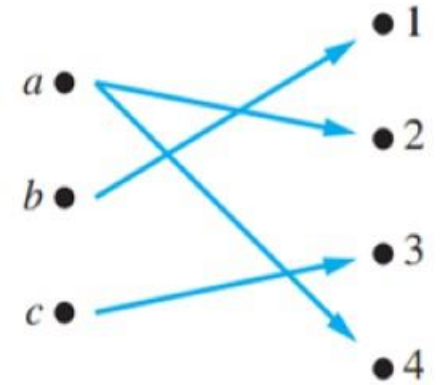
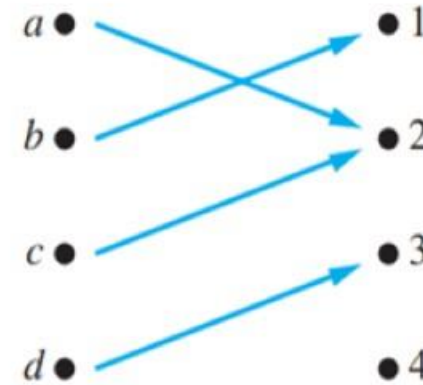
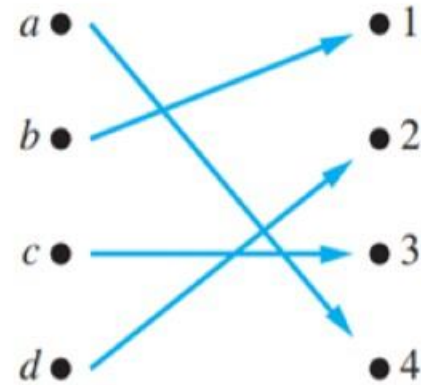
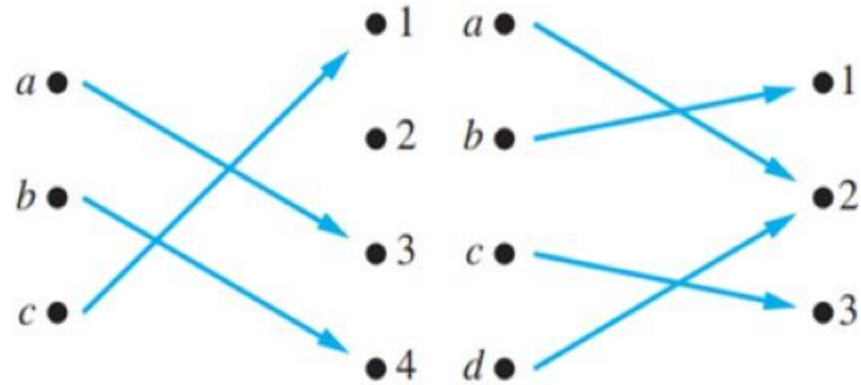
1. True
2. False
3. False

State Type of these Function (one to one / onto / not a function)



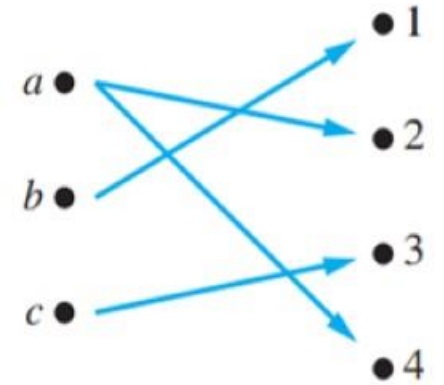
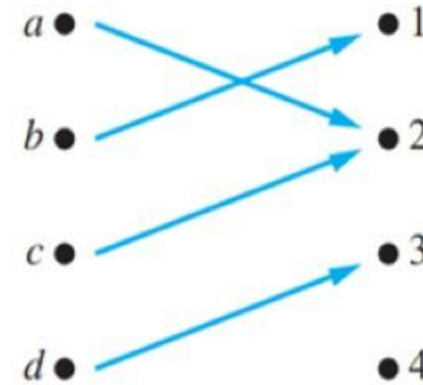
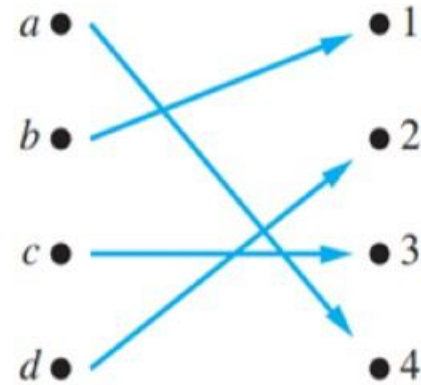
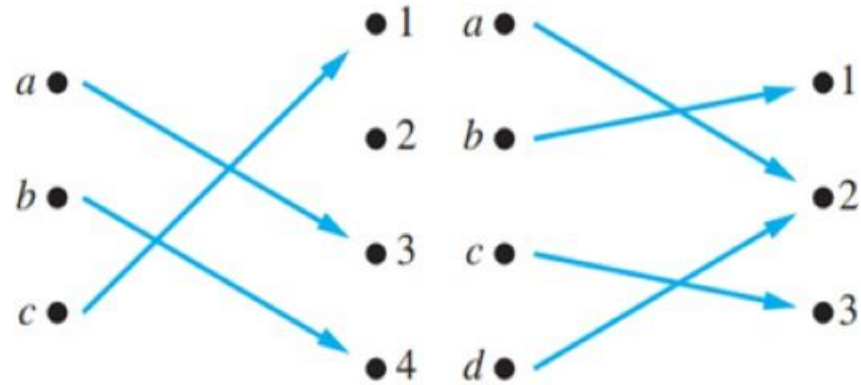
one-to-one + onto $\Rightarrow$ bijection

State Type of these Function (one to one / onto / not a function)



one-to-one + onto $\Rightarrow$ bijection

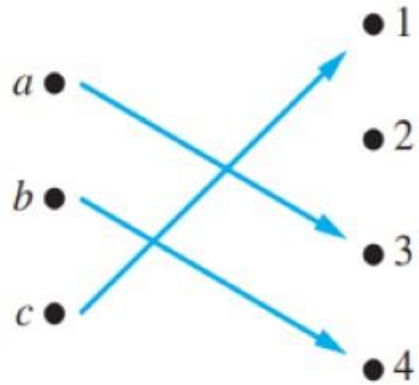
State Type of these Function (one to one / onto / not a function)



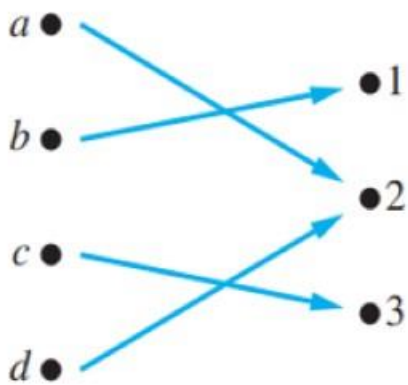
one-to-one + onto $\Rightarrow$ bijection

Solution

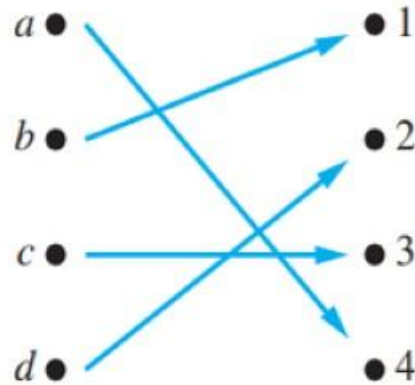
(a) One-to-one,
not onto



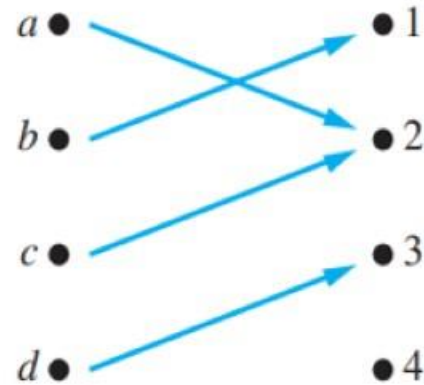
(b) Onto,
not one-to-one



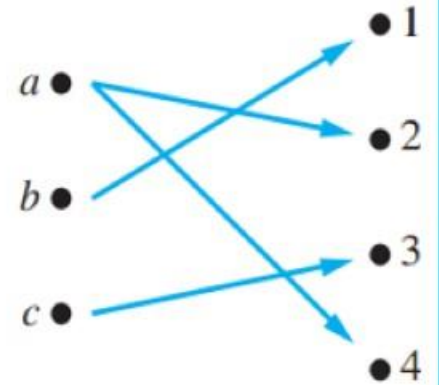
(c) One-to-one
and onto



(d) Neither one-to-one
nor onto

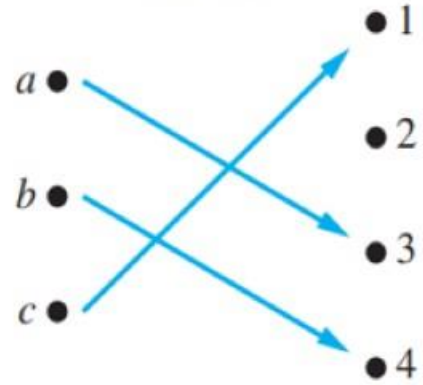


(e) Not a function

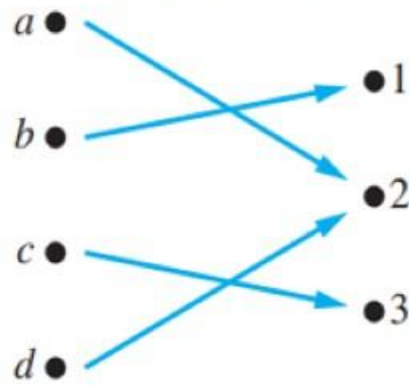


Which of the following is bijective Function(Invertible)?

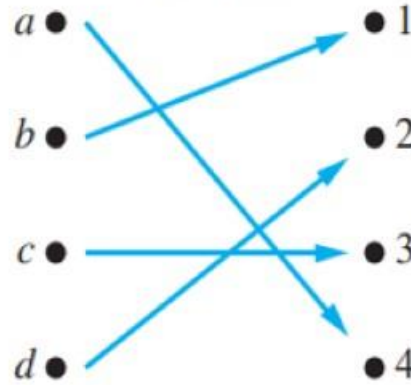
(a) One-to-one,
not onto



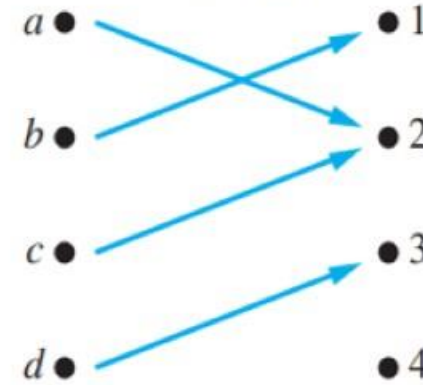
(b) Onto,
not one-to-one



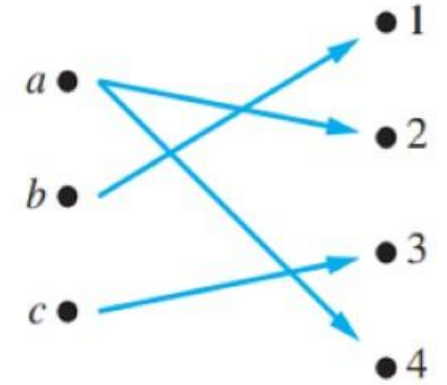
(c) One-to-one
and onto



(d) Neither one-to-one
nor onto



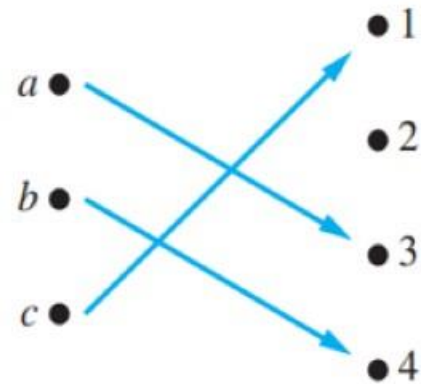
(e) Not a function



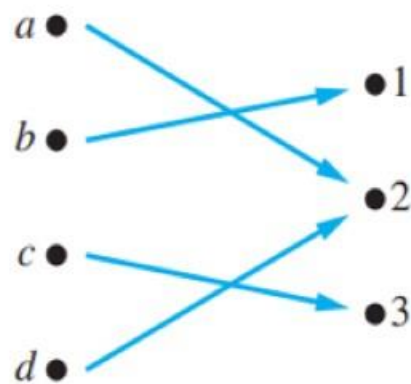
one-to-one + onto $\Rightarrow$ bijection

Solution

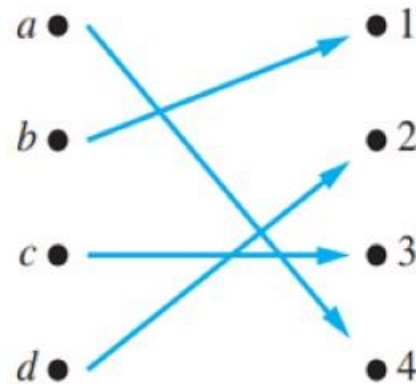
(a) One-to-one,
not onto



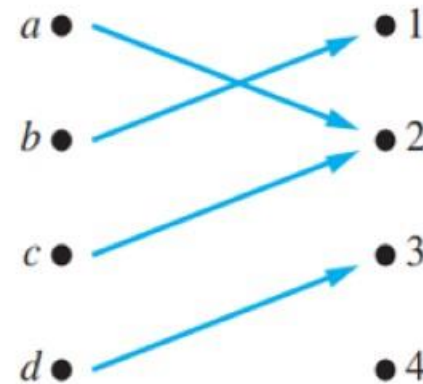
(b) Onto,
not one-to-one



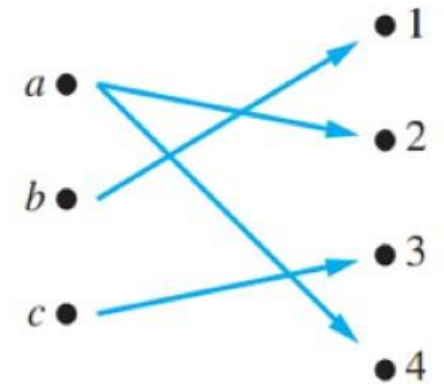
(c) One-to-one
and onto



(d) Neither one-to-one
nor onto



(e) Not a function



Given $f(x) = 2x+3$ and $g(x) = 3x$, find

- $(f \circ g)(x)$

$$6x+3$$

- $(g \circ f)(x)$

$$6x+9$$

- $(g \circ g)(x)$

$$9x$$

- $(f \circ f)(x)$

$$4x+9$$

Given $f: \mathbb{N} \rightarrow \mathbb{Z}^+$

State true or false

- **$0 \in \text{domain}(f)$**
- **$0 \in \text{codomain}(f)$**
- **$-1 \in \text{domain}(f)$**
- **$-1 \in \text{codomain}(f)$**

Solution T,F,F,f

- Definition of a Function.
- Types of function
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial



Thank You