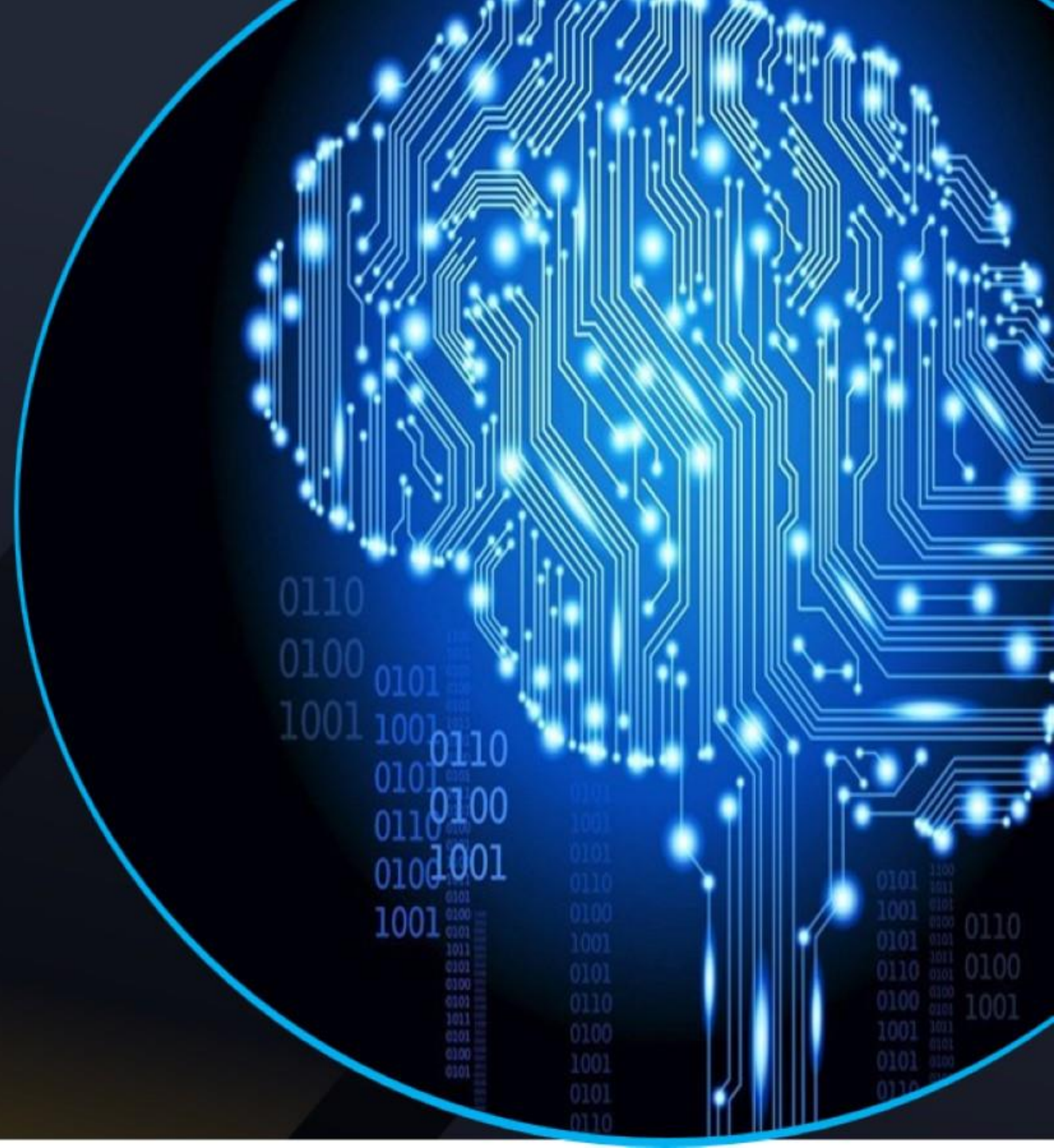


Discrete Mathematics

Dr. Shahira Ahmed

IT241

Lecture 4





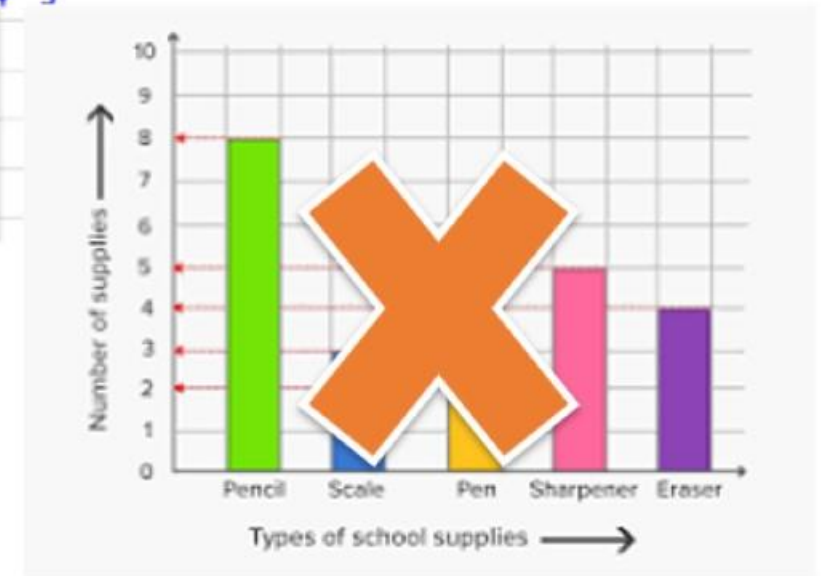
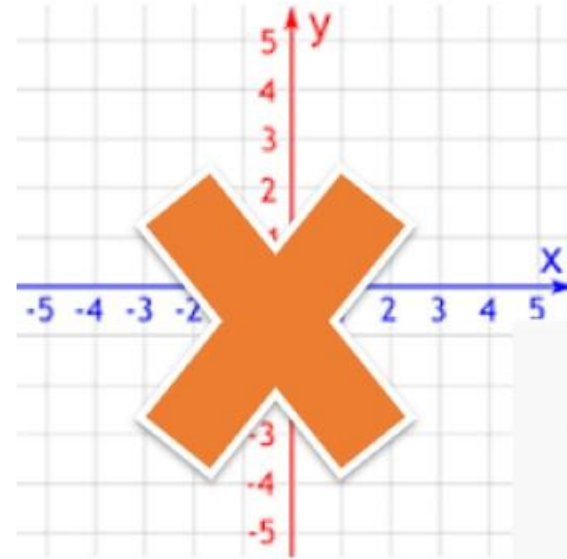
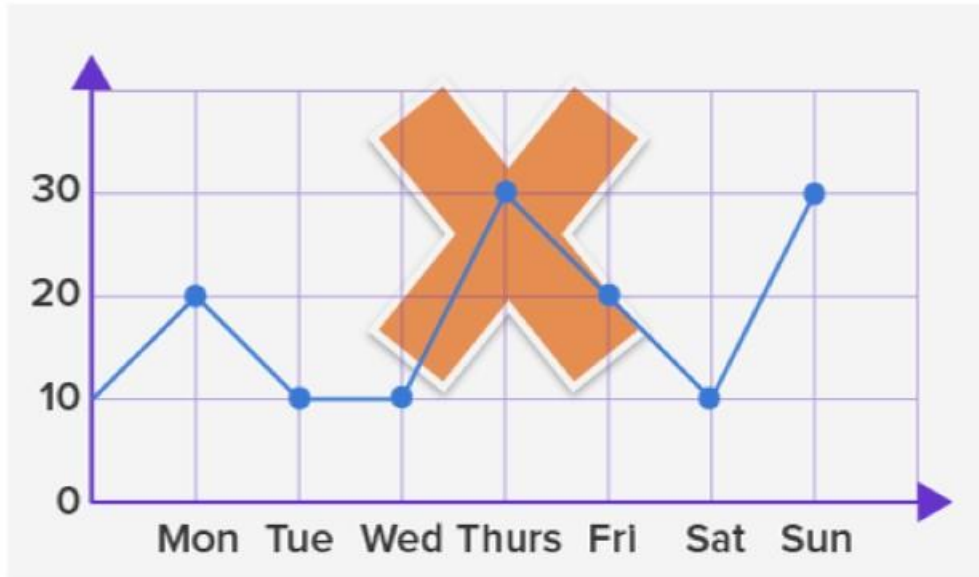
Discrete Mathematics

Lecture 4

Graph Theory

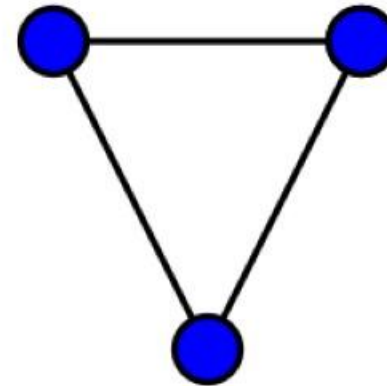
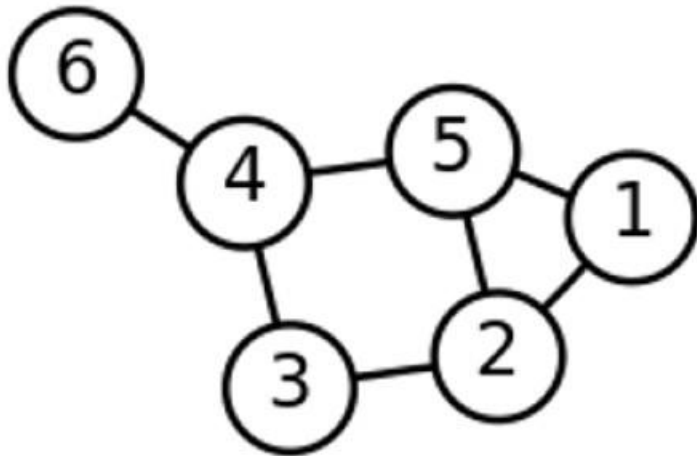
- Graph Definition
- Edge Type
- Graph Type
- Multiplicity
- Degrees
- Neighborhood
- Graph Representation

Not a Graph in discrete math



Not a Graph in discrete math

- Graph in discrete math



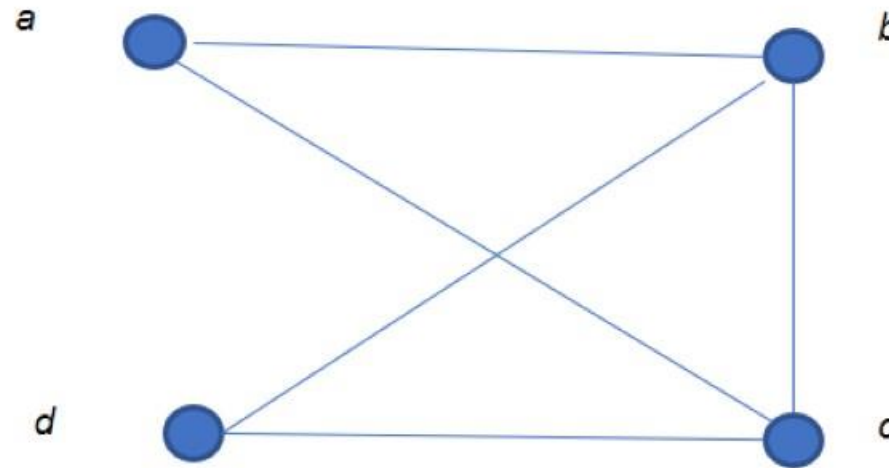
Representation: Graph $G = (V, E)$ consists set of vertices denoted by V , or by $V(G)$ and set of edges E , or $E(G)$.

How many Vertex?

How many Edge?

Solution

- four vertices
- five edges.



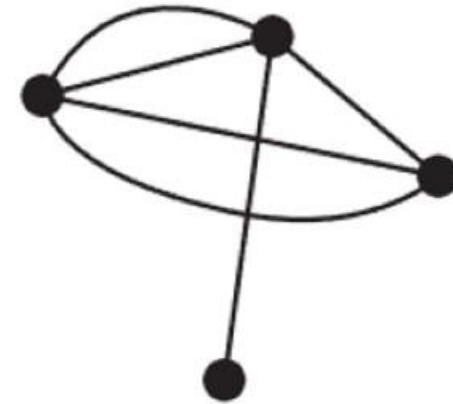
Directed: Ordered pair of vertices. Represented as (u, v) directed from vertex u to v .



Undirected: Unordered pair of vertices. Represented as $\{u, v\}$. Disregards any sense of direction and treats both end vertices interchangeably.

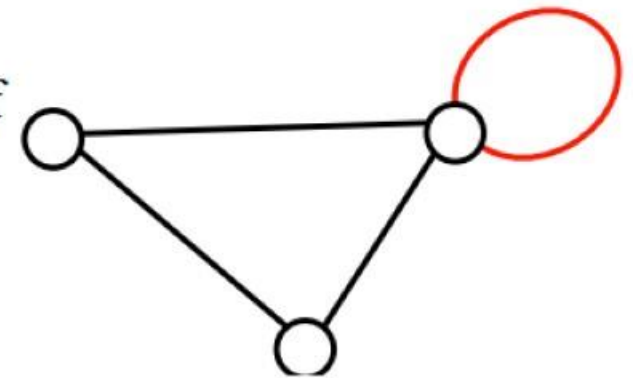


Multiple Edges: Two or more edges joining the same pair of vertices.

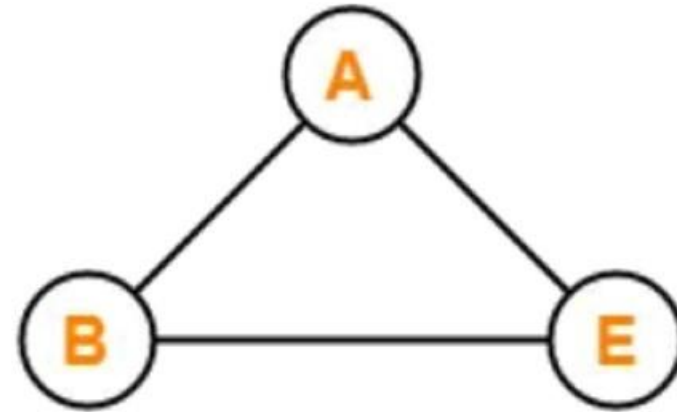


Loop: An edge that connects a vertex to itself

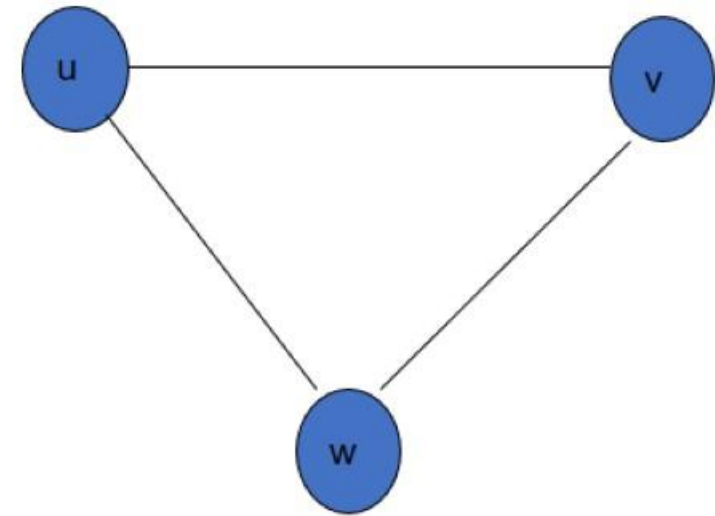
a loop (also called a self-loop) is an edge that connects a vertex to itself



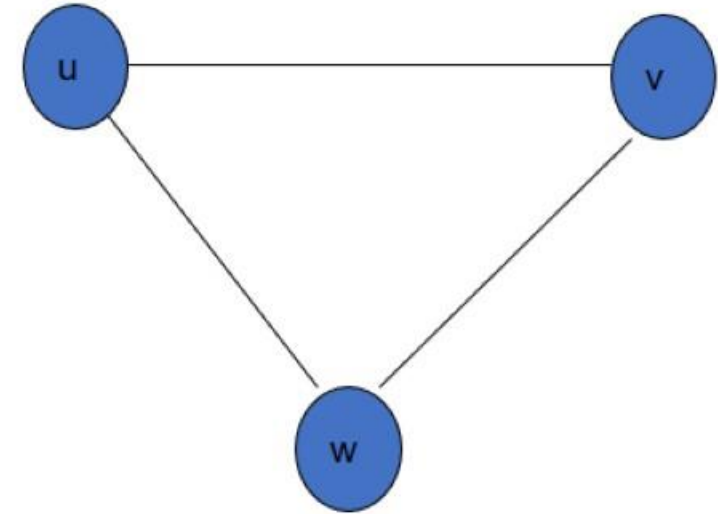
- A **Simple** graph have
 - no self loops
 - no multiple edges.



Simple (Undirected) Graph: consists of V , a nonempty set of vertices, and E , a set of **unordered pairs** of distinct elements of V called edges (**undirected**)

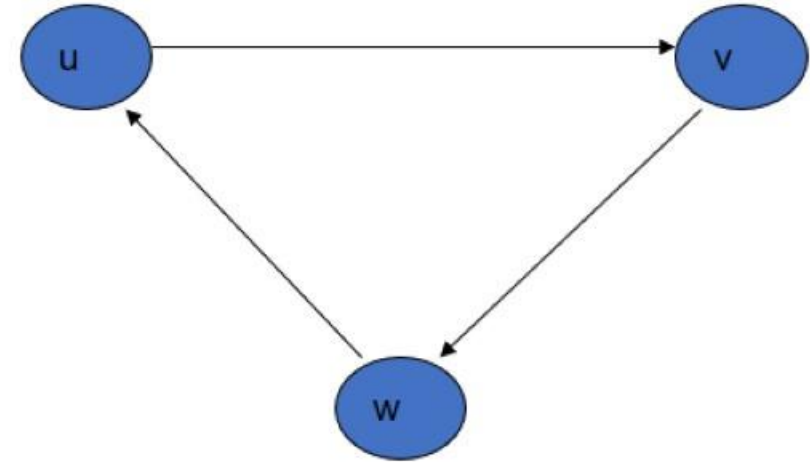


Represent graph in the opposite figure

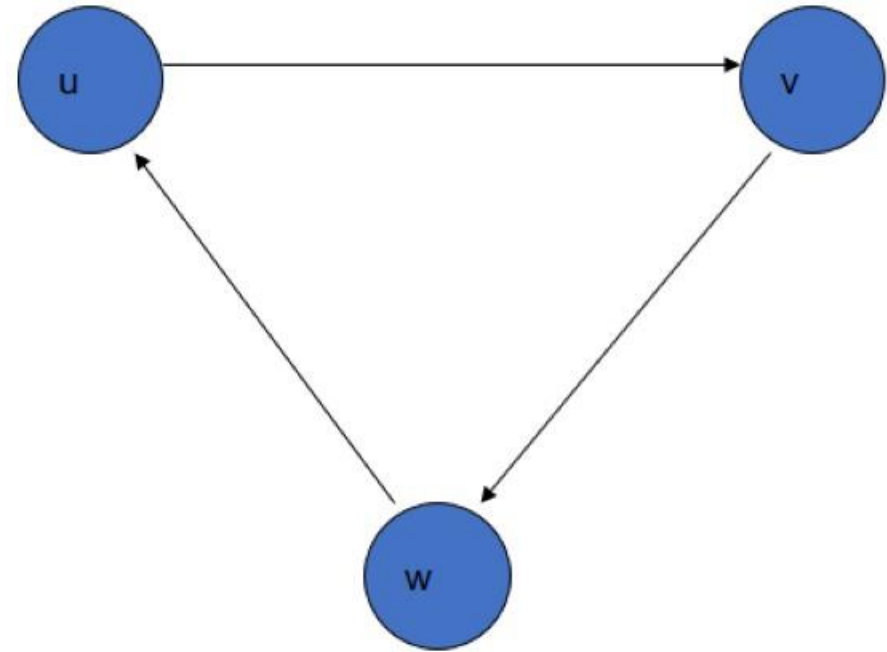


$$G(V, E), V = \{u, v, w\}, E = \{\{u, v\}, \{v, w\}, \{u, w\}\}$$

Simple Directed Graph: $G(V, E)$, set of vertices V , and set of Edges E , that are **ordered pair** of elements of V (directed edges)

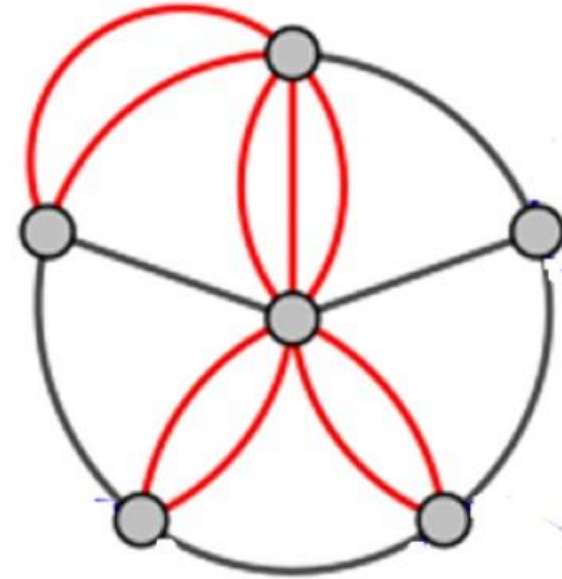


Represent Graph?



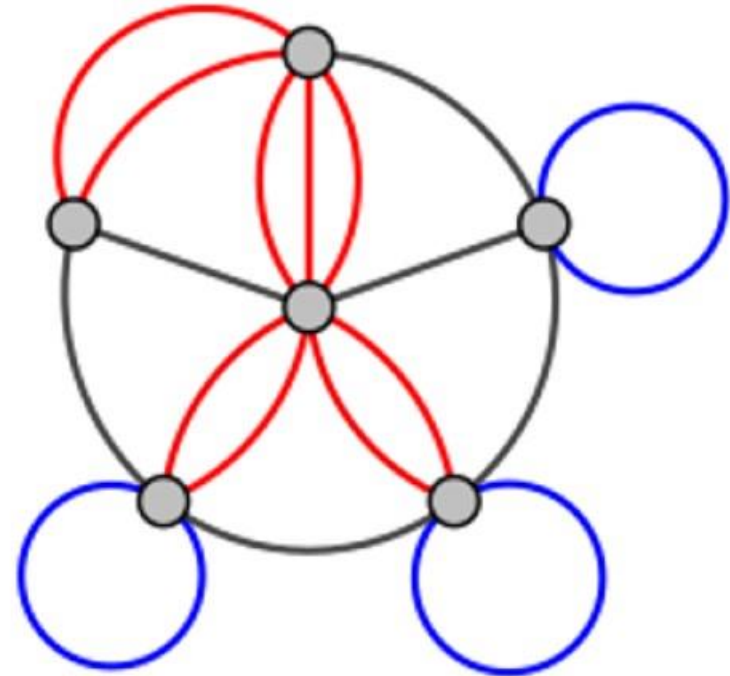
$$G(V, E), V = \{u, v, w\}, E = \{(u, v), (v, w), (w, u)\}$$

Multigraph: have **multiple** (parallel edges)



Pseudograph:

- **Loops** allowed in such a graph
- **Multiple** edges.



- A null graph is a graph in which there are **no edges** between its vertices.
- Edge set is **empty**



- A trivial graph is the graph which has only one vertex.

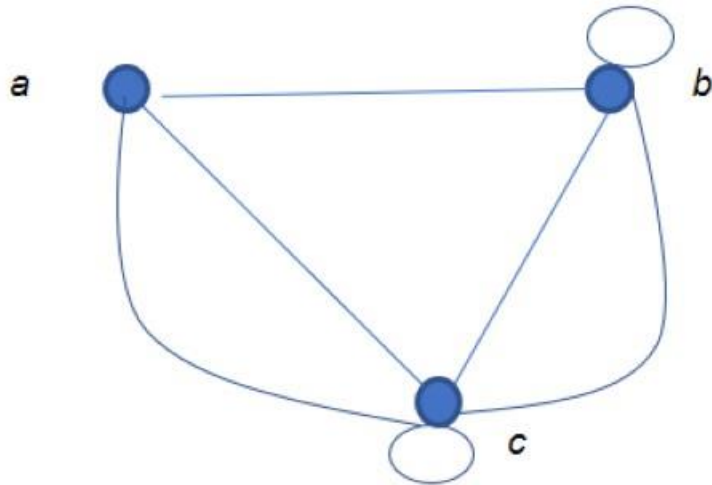
• **v**

- Match

Trivial Graph

Null Graph

Pseudo Graph



- Solution

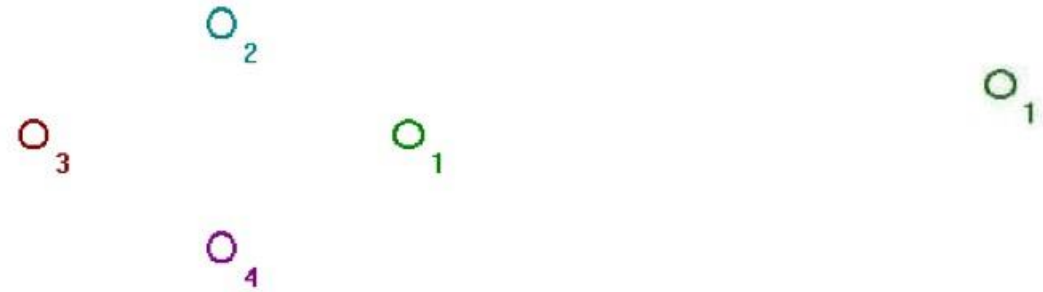
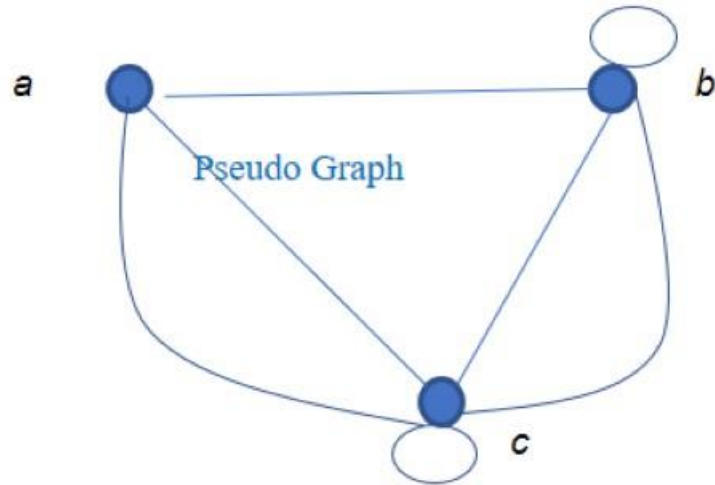
- A. Pseudo Graph
- B. Null Graph
- C. Trivial Graph

- Match

Trivial Graph

Null Graph

Pseudo Graph

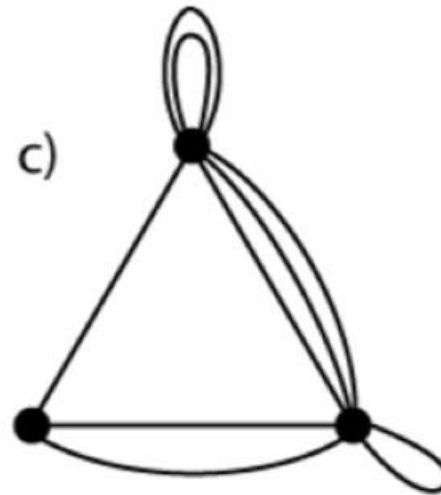
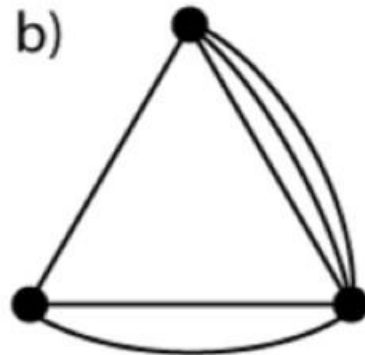
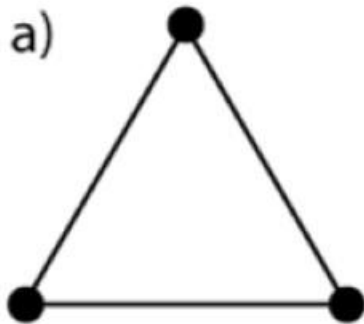


- Match

Simple Graph

multigraph

Pseudo Graph

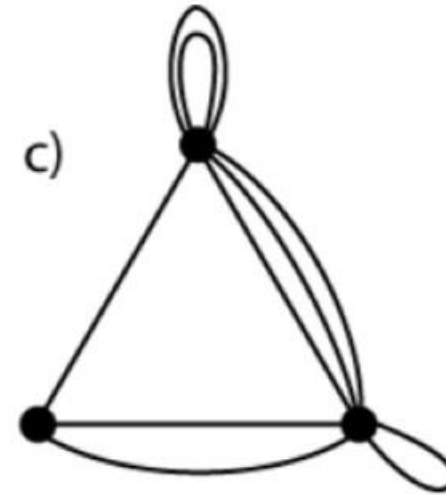
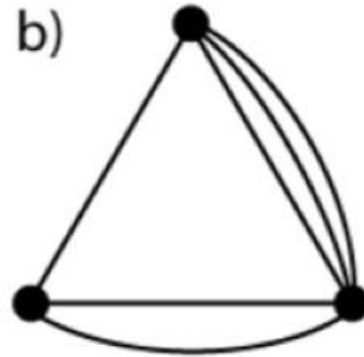
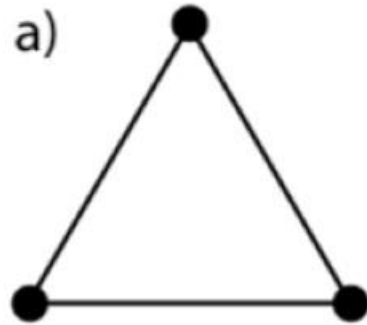


- Match

Simple Graph

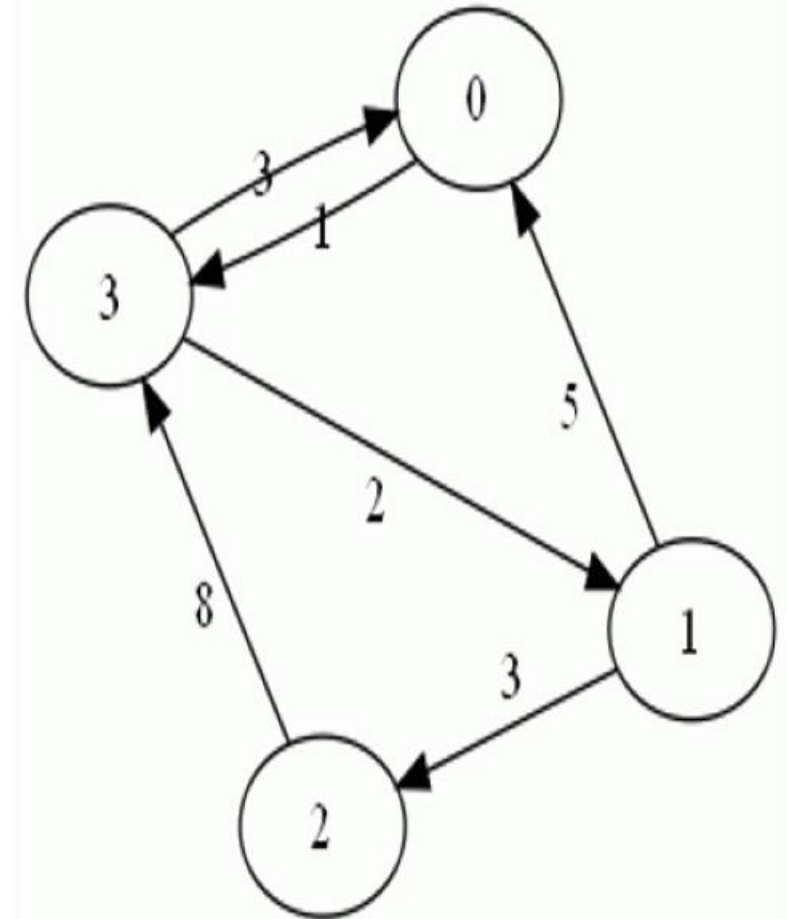
multigraph

Pseudo Graph

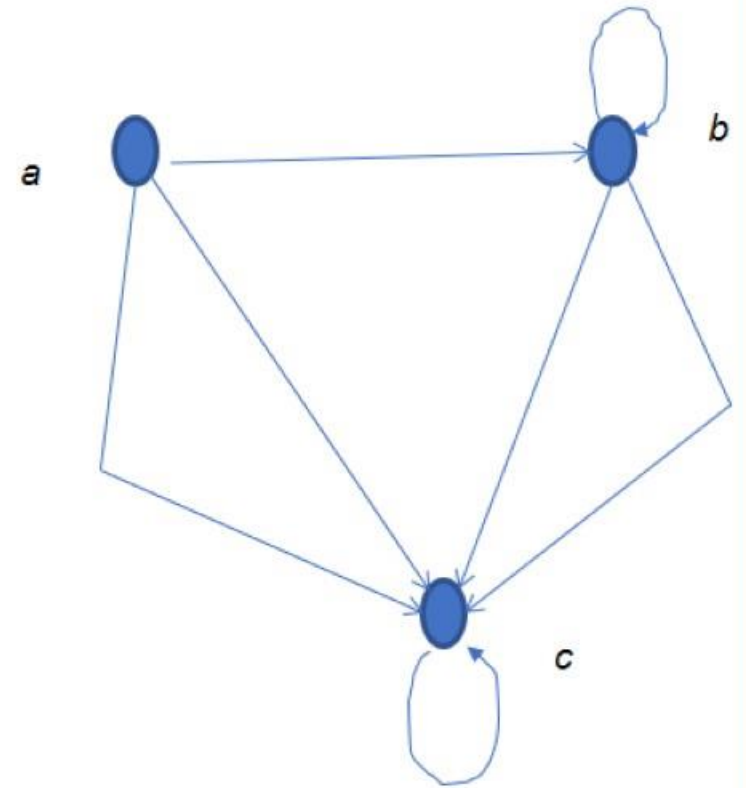


- **Edges** → undirected or directed (or both-Mixed)?
- **Multiple** edges → multigraph?
- **Loops** → Pseudo

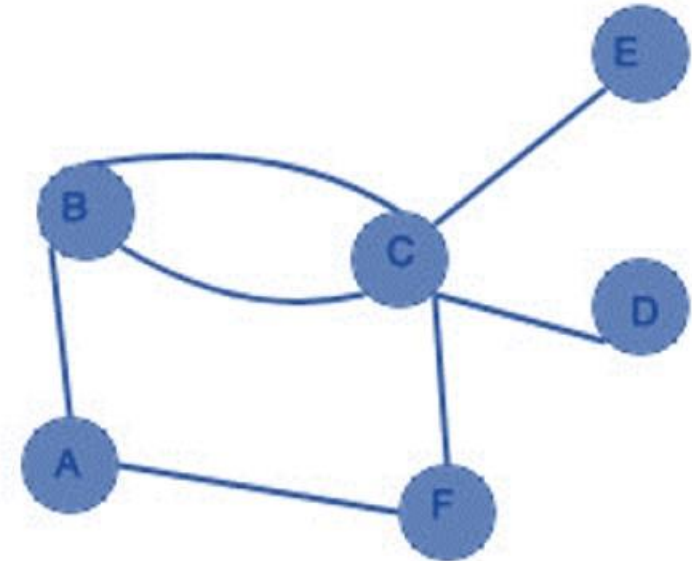
- *Simple directed graph*
 - has no loops and no multiple edges.



- *Pseudo graph*



- **Multiplicity** : the **number of edges** between two vertices.
- The multiplicity of the edge $\{B, C\}$?
- The multiplicity of the edge $\{B, C\}=2$



- **Find**

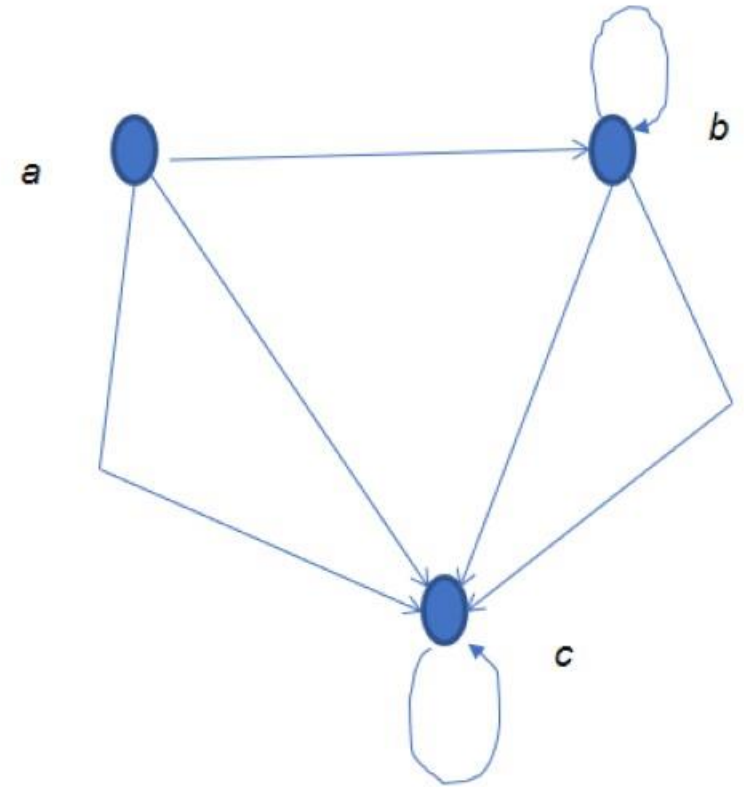
- **Multiplicity of (a,b)**
- **Multiplicity of (b,c)**

Multiplicity

- Since there can be multiple edges between the same pair of vertices, the **multiplicity** of edge tells the number of edges between the same pair of vertices.
- The multiplicity of the edge {B, C} is 2

- **Solution**

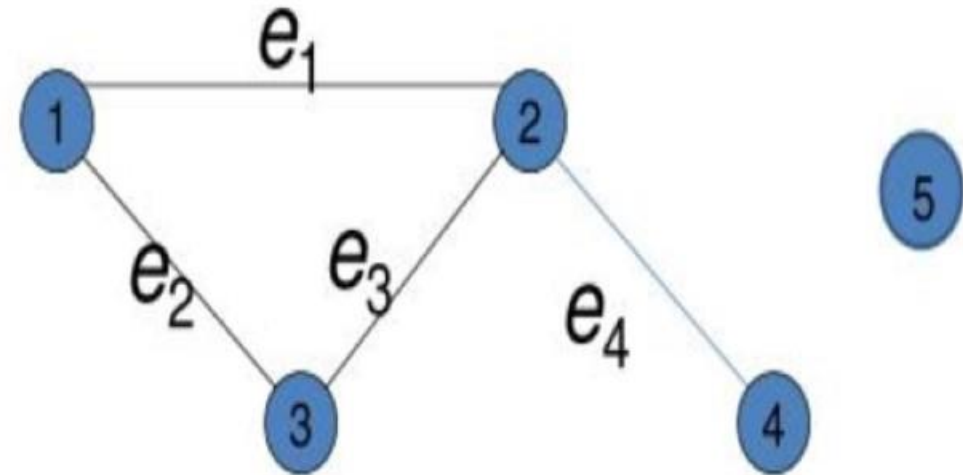
- **Multiplicity of (a,b) is 1**
- **Multiplicity of (b,c) is 2**



Adjacent (or *Neighbors*) in G if there is an edge e between u and v

***Adjacent* (or *Neighbors*)** in G if there is an edge e between u and v

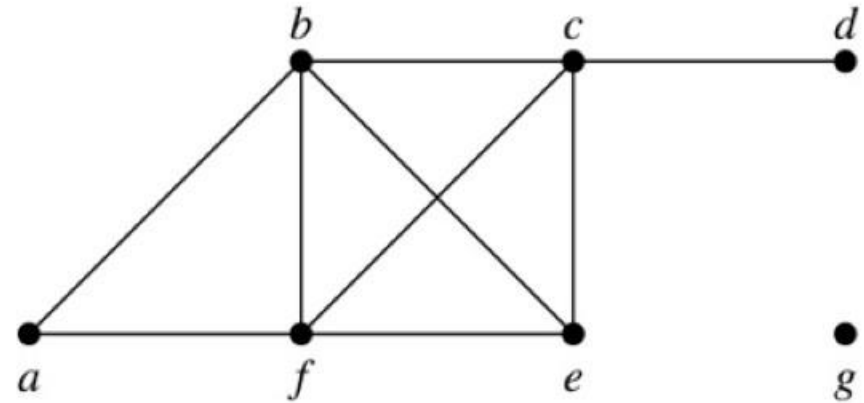
- 1 is adjacent to 2,3
- 2 is adjacent to 1, 3, and 4
- 3 is adjacent to 1 and 2
- 4 is adjacent to 2
- 5 is not adjacent to any vertex



The *degree of a vertex in a undirected graph* is the number of edges incident with it,

The degree of the vertex v is denoted by $\deg(v)$.

Example: What are the degrees of the vertices in the graphs G ?

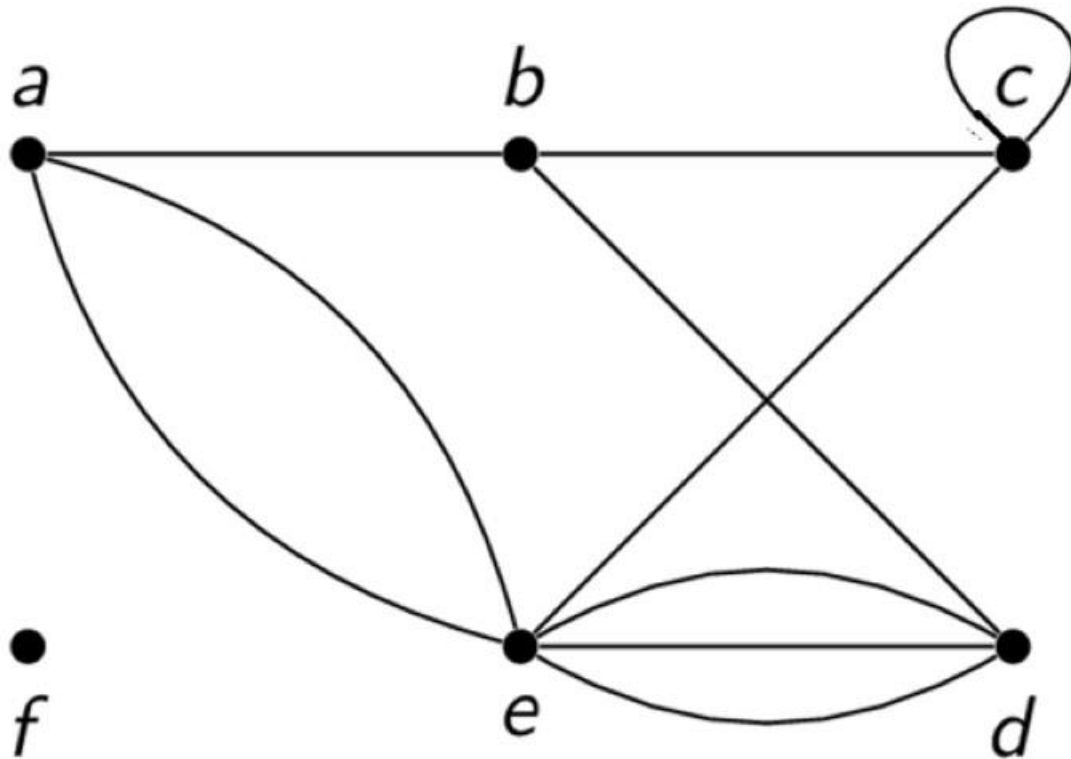


Solution:

G : $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$,
 $\deg(e) = 3$, $\deg(g) = 0$.

Degree of a vertex

- A **loop** at a vertex contributes **two** to the degree of that vertex.



$$\deg a = 3$$

$$\deg b = 3$$

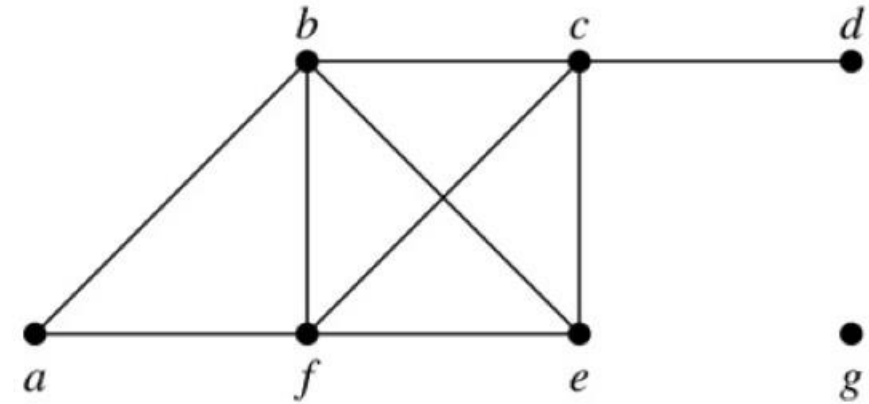
$$\deg c = 4$$

$$\deg d = 4$$

$$\deg e = 6$$

$$\deg f = 0$$

What are neighborhoods of the vertices in the graphs G ?



Solution:

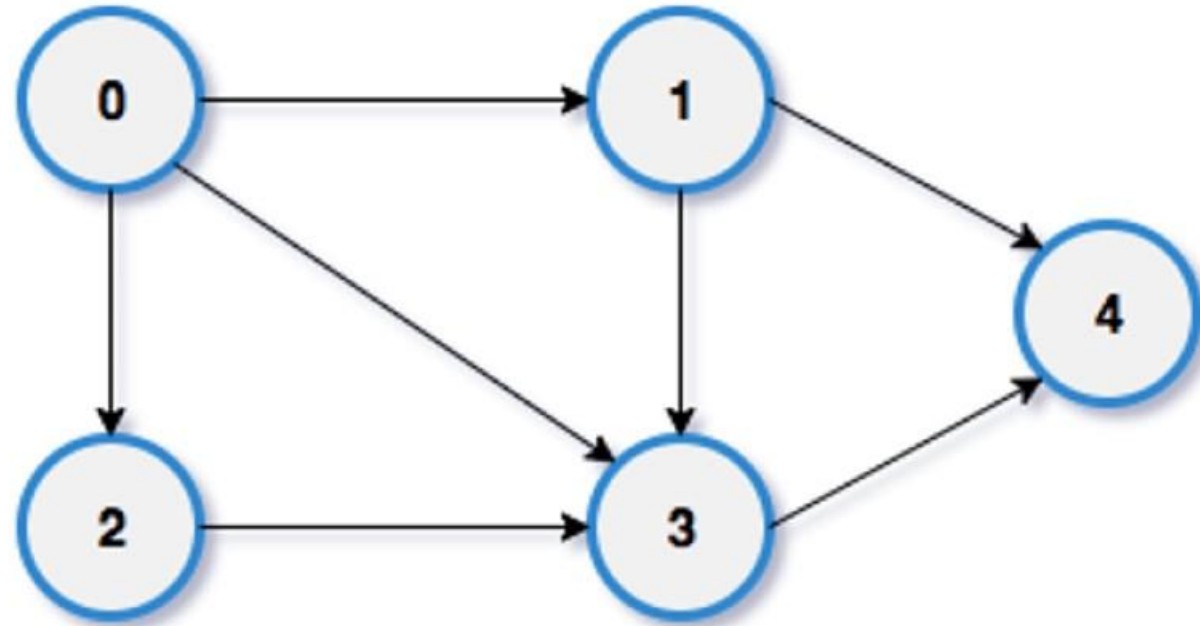
G :

$$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, N(c) = \{b, d, e, f\}, N(d) = \{c\}, \\ N(e) = \{b, c, f\}, N(f) = \{a, b, c, e\}, N(g) = \emptyset.$$

- The *in-degree* of a vertex v , denoted $\deg^-(v)$, is the number of edges which terminate at v .
- The *out-degree* of v , denoted $\deg^+(v)$, is the number of edges with v as their initial vertex.
- **Loop** at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

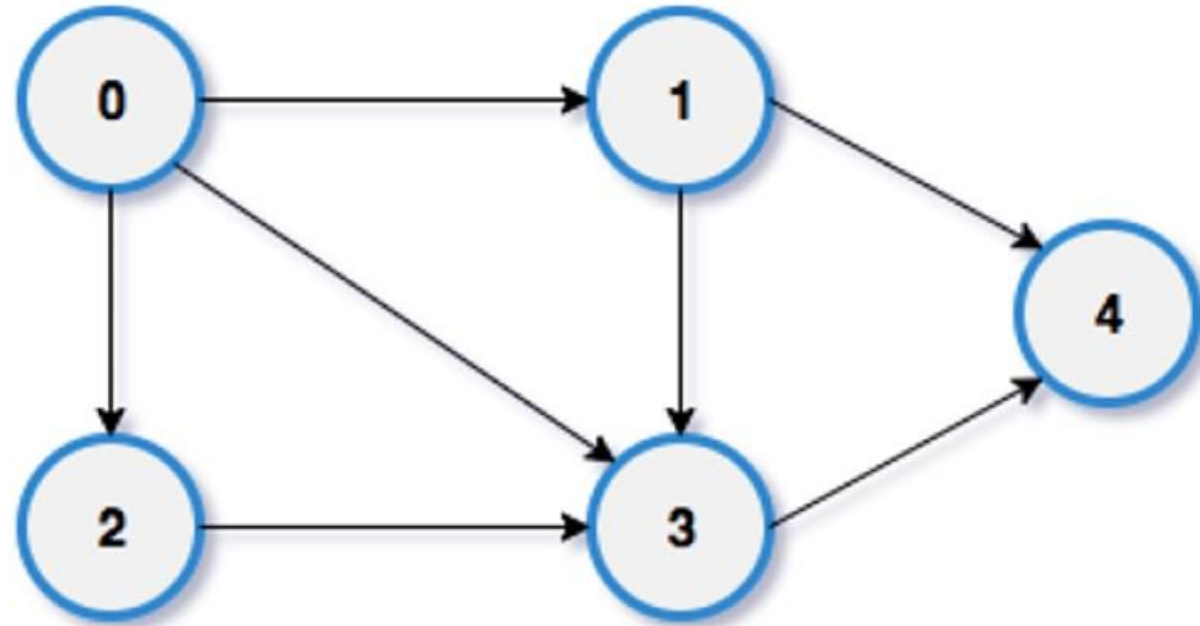
In-degree:

- $\deg^-(0) = 0$
- $\deg^-(1) = 1$
- $\deg^-(2) = 1$
- $\deg^-(3) = 3$
- $\deg^-(4) = 2$

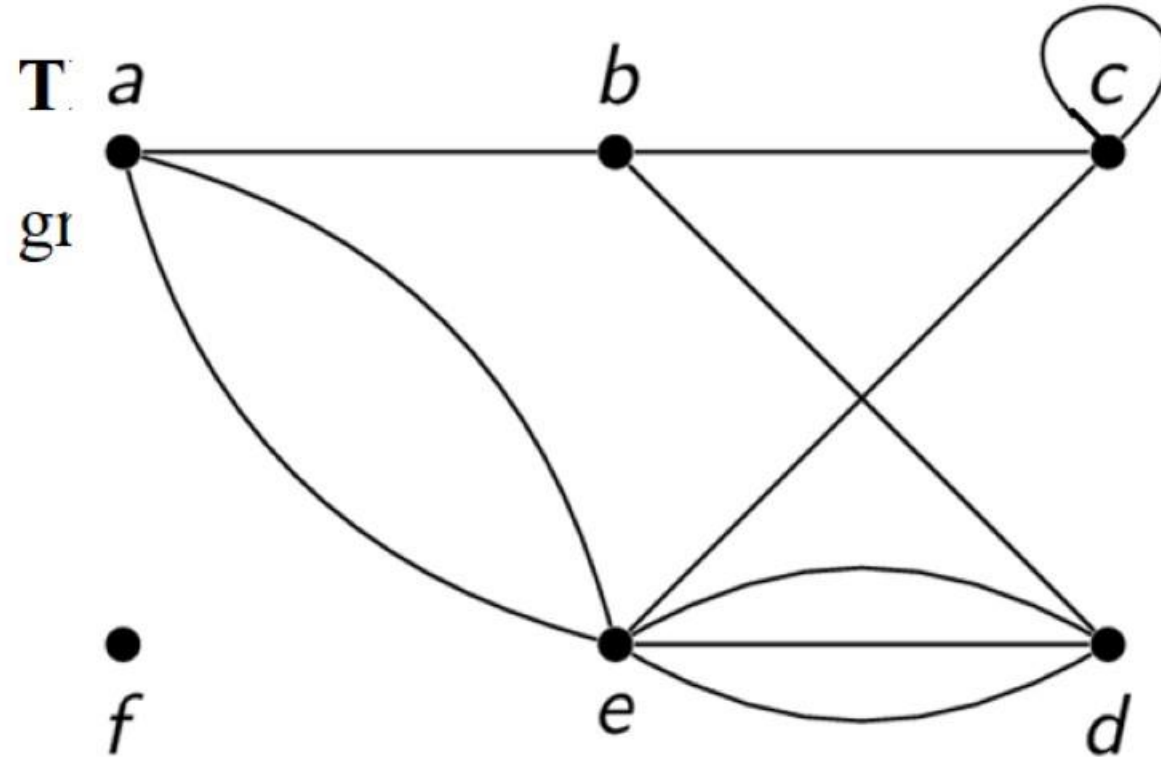


out-degree:

- $\deg^+(0) = 3$
- $\deg^+(1) = 2$
- $\deg^+(2) = 1$
- $\deg^+(3) = 1$
- $\deg^+(4) = 0$



Degrees of Vertices



If $G = (V, E)$ is an undirected

$$\deg a = 3$$

$$\deg b = 3$$

$$\deg c = 4$$

$$\deg d = 4$$

$$\deg e = 6$$

$$\deg f = 0$$

$\deg(v)$

sum of degrees

How many edges are there in a graph with 10 vertices of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, the handshaking theorem tells us that $2m = 60$.

So the number of edges $m = 30$.

A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

Solution:

$$2 * \text{number of edges} = \text{sum of degrees}$$

$$2 * 24 = \text{sum of degrees}$$

$$48 = \text{number of vertex} * 4$$

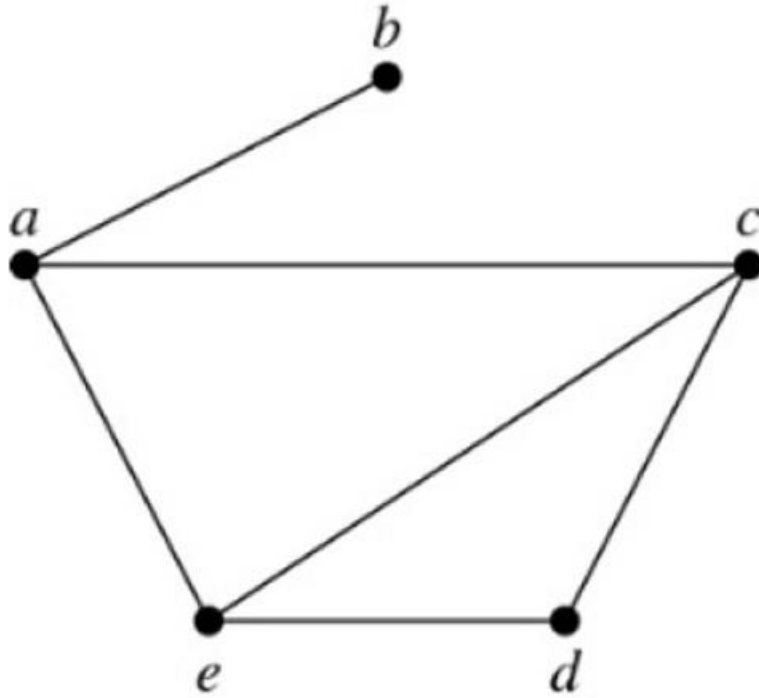
$$\text{number of vertex} = 12$$

Graph Representation

Dr. Shahira Ahmed

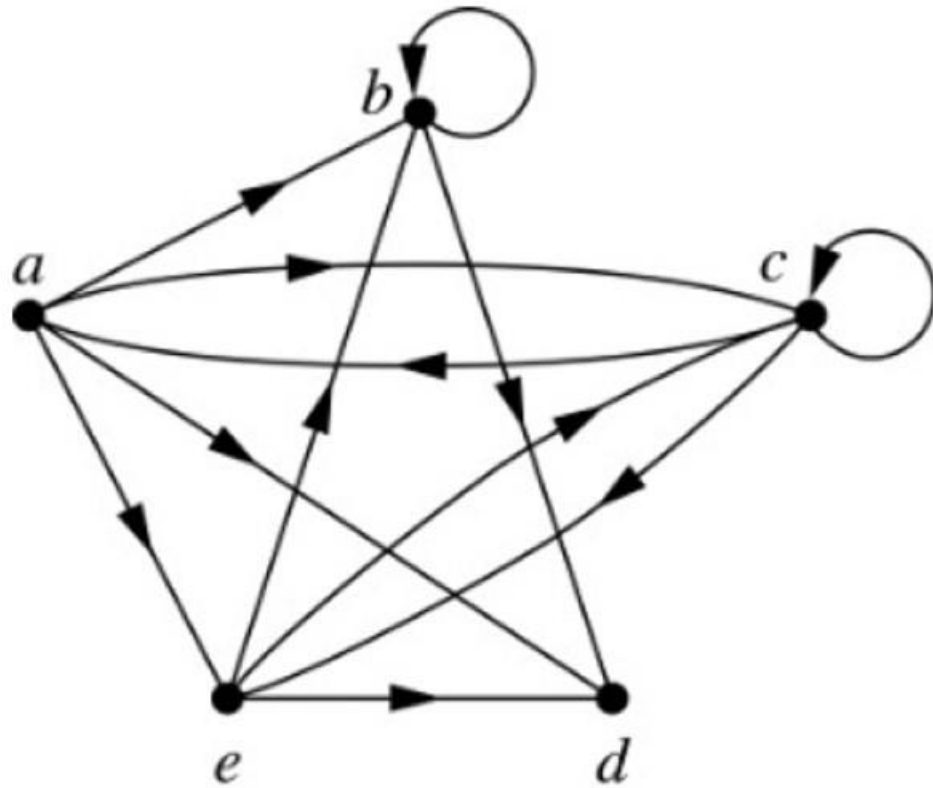
Definition: An *adjacency list* can be used to represent a graph with **no multiple edges** by specifying the vertices that are adjacent to each vertex of the graph.

Representing Graphs: Adjacency Lists



<i>Vertex</i>	<i>Adjacent Vertices</i>
<i>a</i>	<i>b, c, e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a, d, e</i>
<i>d</i>	<i>c, e</i>
<i>e</i>	<i>a, c, d</i>

Representing Graphs: Adjacency Lists



<i>Initial Vertex</i>	<i>Terminal Vertices</i>
<i>a</i>	<i>b, c, d, e</i>
<i>b</i>	<i>b, d</i>
<i>c</i>	<i>a, c, e</i>
<i>d</i>	
<i>e</i>	<i>b, c, d</i>

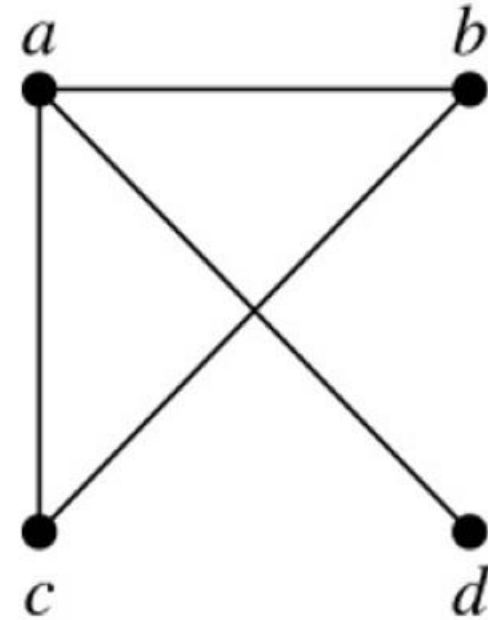
Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. The *adjacency matrix* $\mathbf{A}_G$ of G the $n \times n$ zero-one matrix with:

- 1 as its (i, j) th entry when v_i and v_j are adjacent,
- 0 as its (i, j) th entry when they are not adjacent.
- In other words, if the graphs adjacency matrix is

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

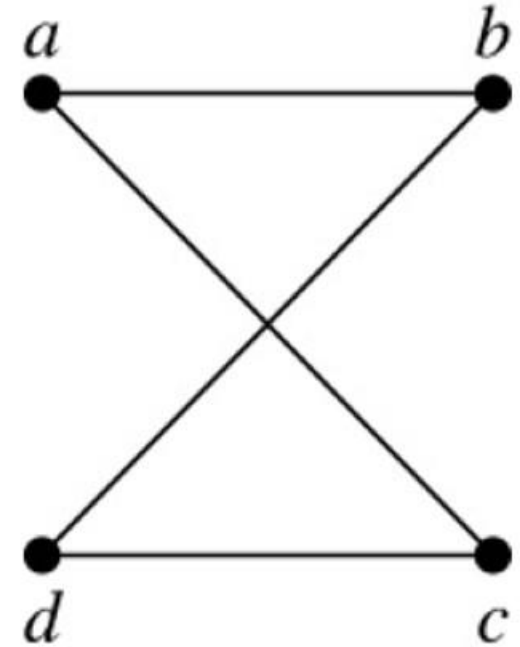
Represent the Following Graph using Adjacency Matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

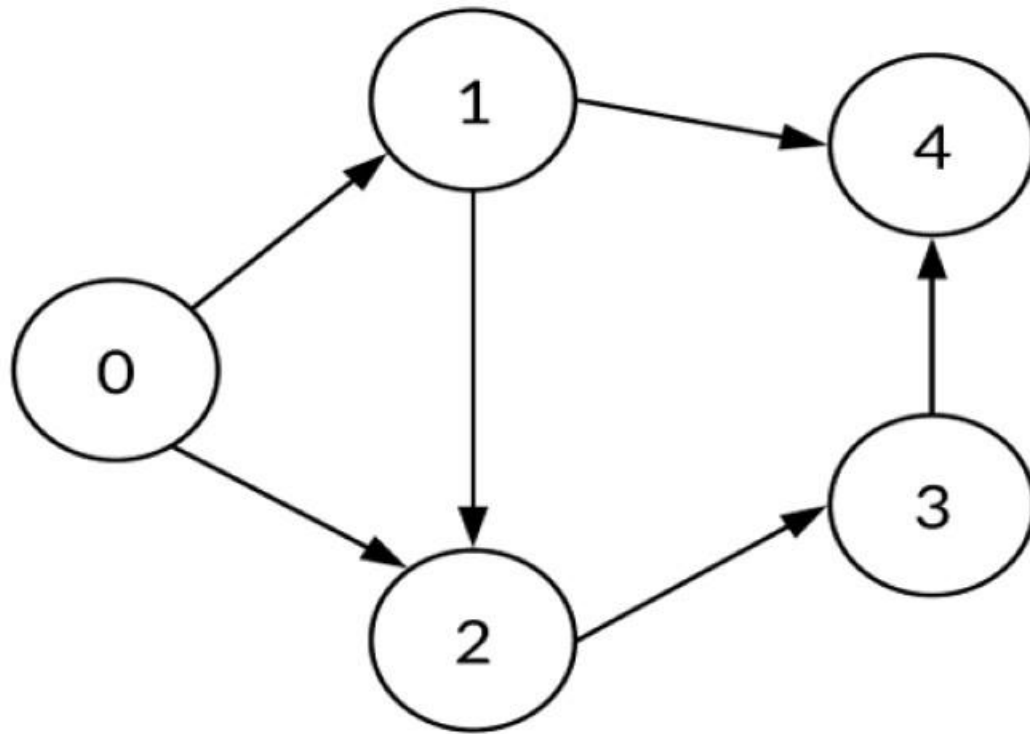


- Represent the Following Graph using Adjacency Matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Represent the Following Graph using Adjacency Matrix

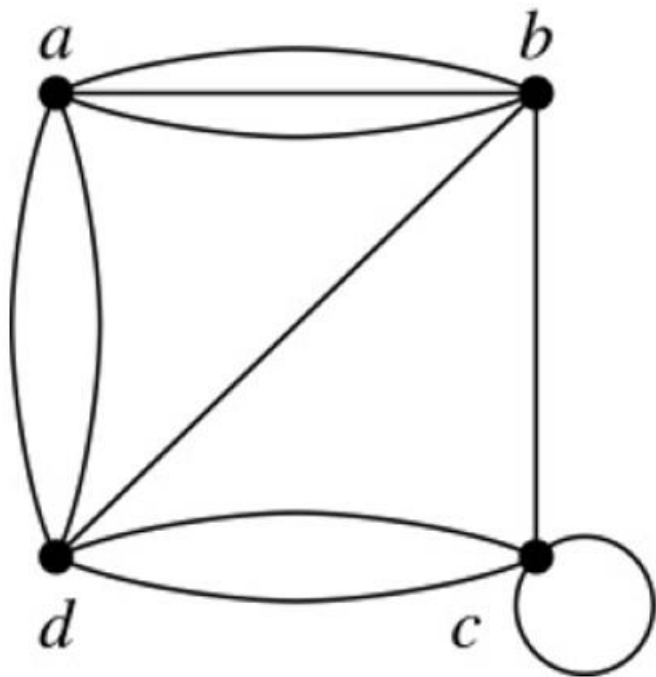


Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

- Adjacency matrices can also be used to represent graphs with loops and multiple edges.
- When **multiple** edges connect the same pair of vertices or if multiple **loops** are present at the same vertex , the (i, j) th entry equals the number of edges connecting the pair of vertices.

Example: We give the adjacency matrix of the pseudograph shown here using the ordering of vertices a, b, c, d .



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

- Graph Type

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

- Graph Type

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Multiedge
Loops

undirected

Adjacency Matrix

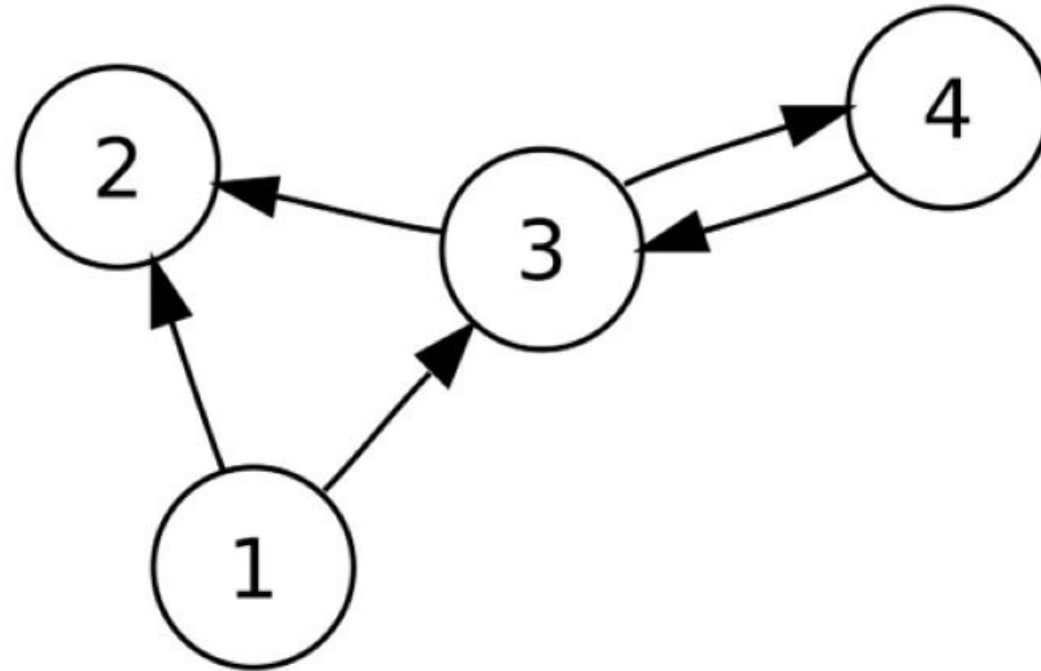
	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

Directed

Recap

- True or False

- $(1,2) \in G$
- $(3,2) \in G$
- $(2,2) \in G$
- $(2,1) \in G$

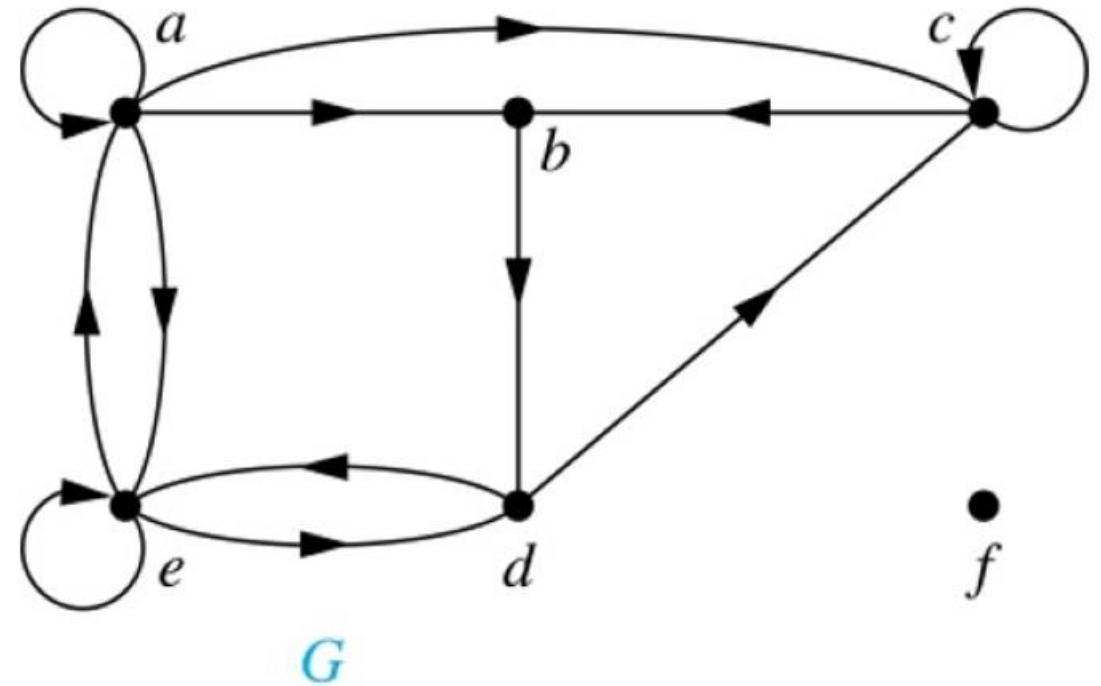


Solution

True, True, False, False

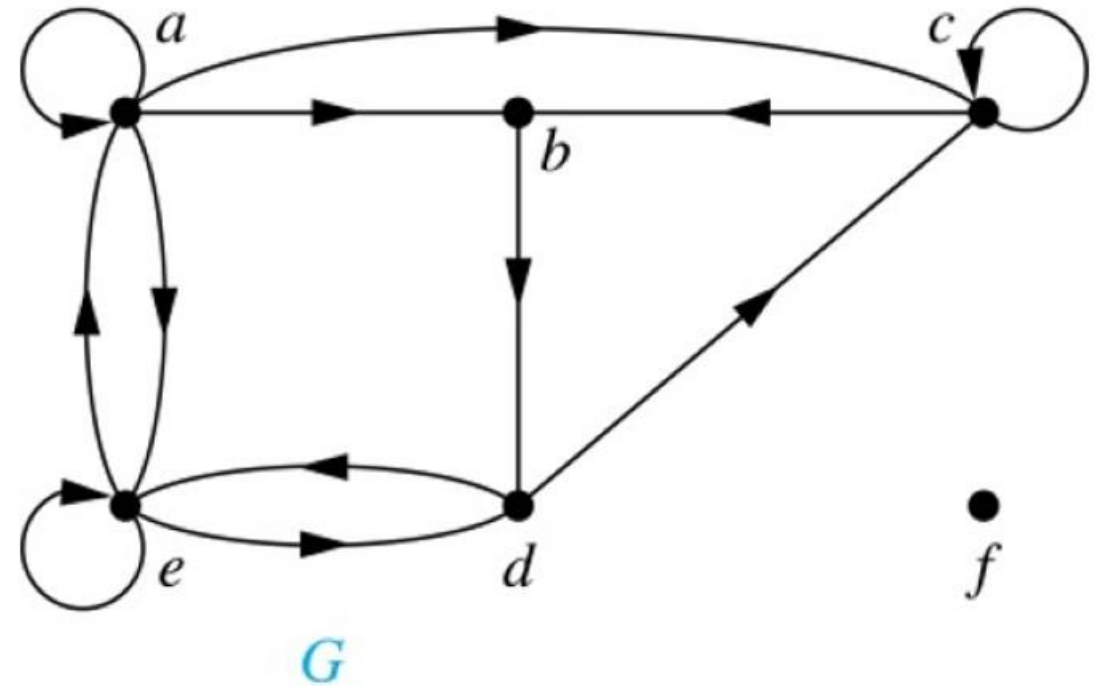
In the graph *Find all indegree and out degree for all nodes*

$\deg^-(a) = 2$	$\deg^+(a) = 4$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2,$	$\deg^+(d) = 2$
$\deg^-(e) = 3$	$\deg^+(e) = 3$
$\deg^-(f) = 0.$	$\deg^+(f) = 0.$

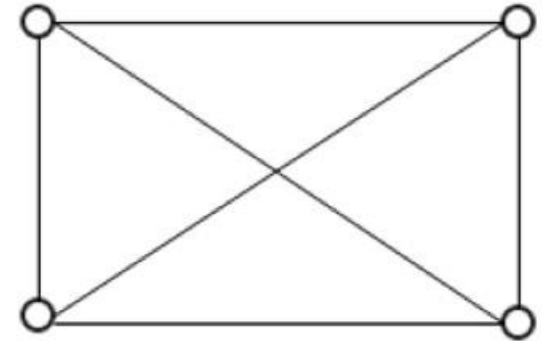
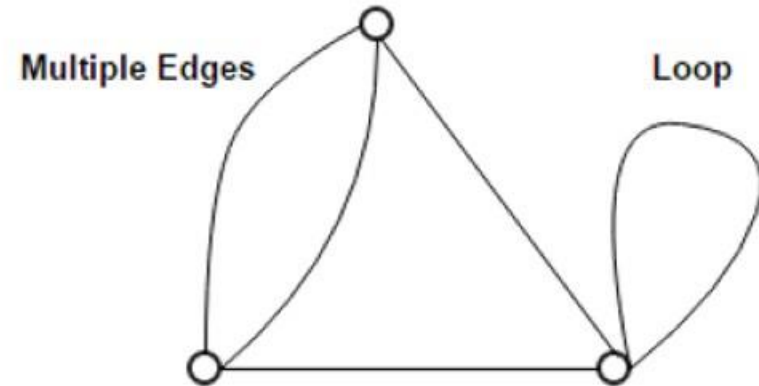


In the graph *Find all indegree and out degree for all nodes*

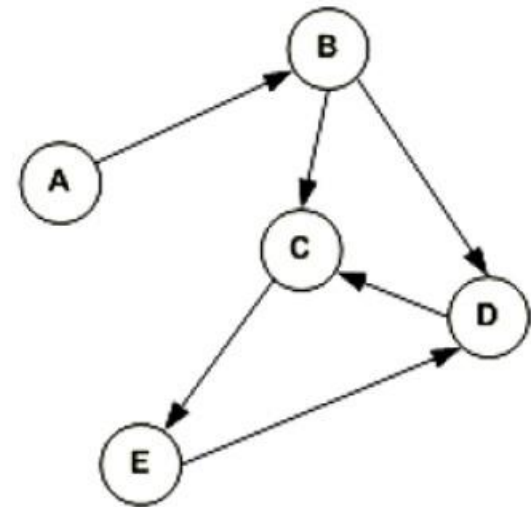
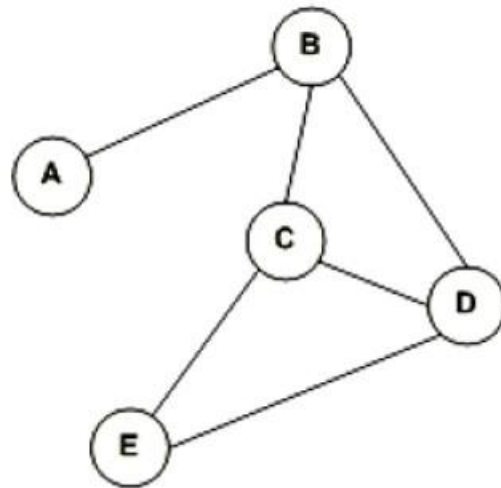
$\deg^-(a) = 2$	$\deg^+(a) = 4$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2,$	$\deg^+(d) = 2$
$\deg^-(e) = 3$	$\deg^+(e) = 3$
$\deg^-(f) = 0.$	$\deg^+(f) = 0.$



Simple Graph or not?

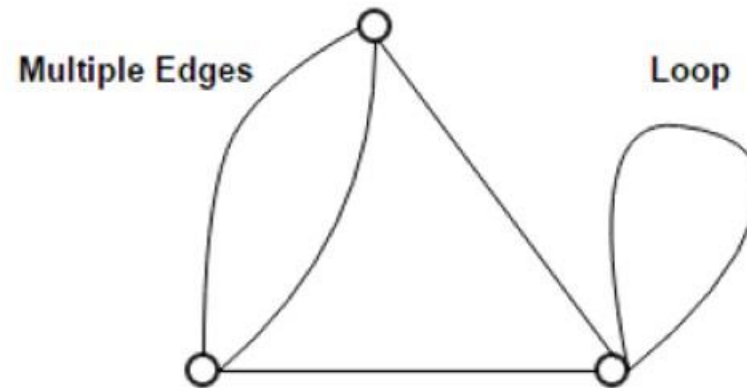


Directed or undirected ?

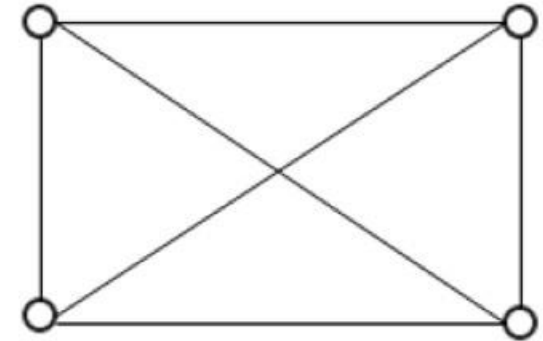


Simple Graph or not?

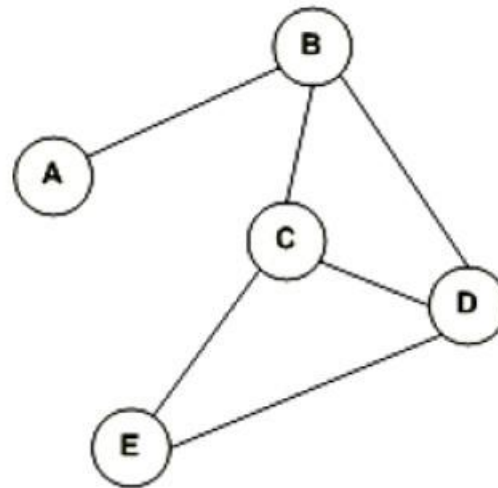
not simple graph



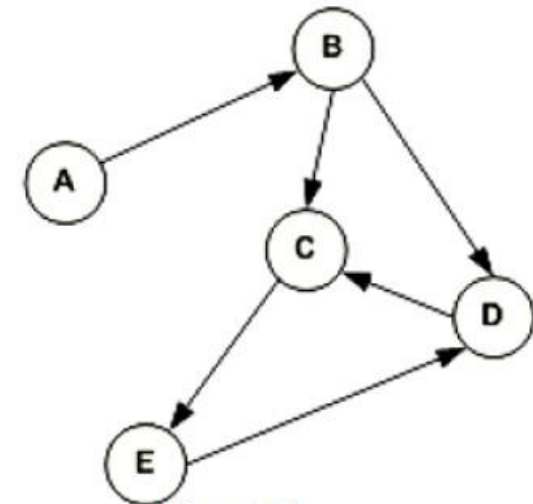
Simple Graph



Directed or undirected ?

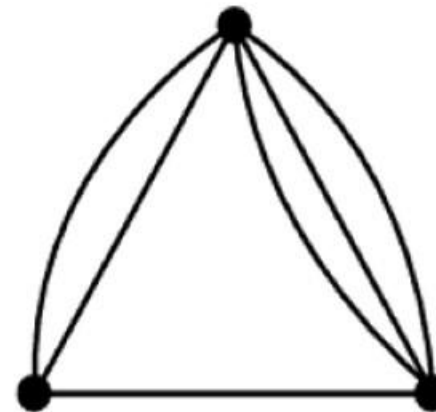
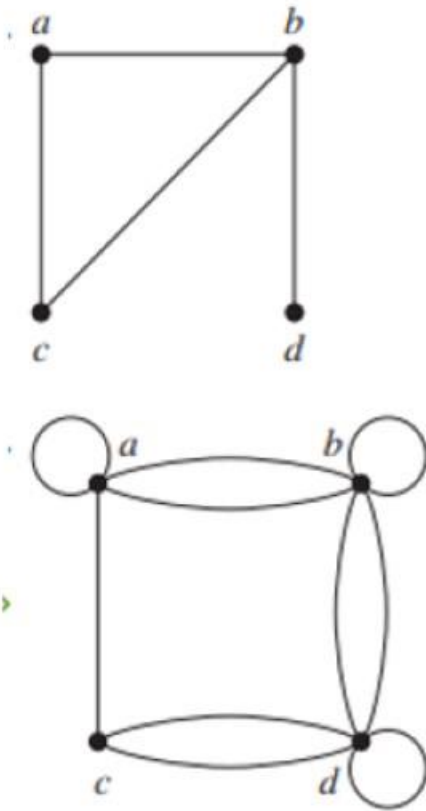


Undirected



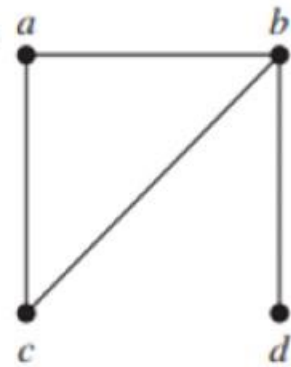
Directed

- State type of graph (Simple-multigraph-pseudograph)

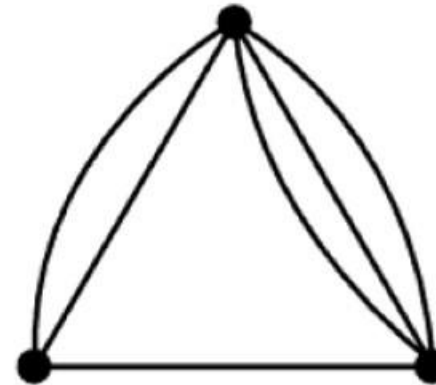
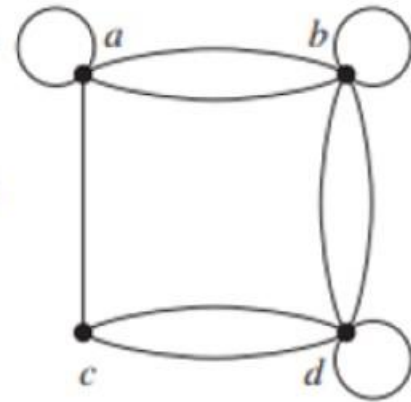


State the type of graph in the following figures

Simple
graph

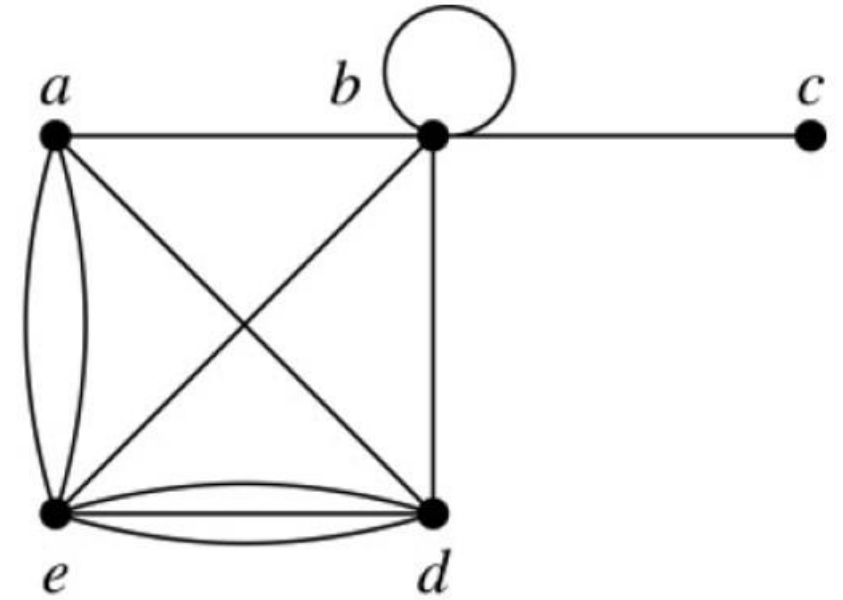


Pseudo
graph



multigraph

Example: What are the degrees and neighborhoods of the vertices in the graphs H ?



Solution:

$$\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1,$$

$$\deg(d) = 5.$$

$$N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, N(c) = \{b\},$$

$$N(d) = \{a, b, e\}, N(e) = \{a, b, d\}.$$

H

How many edges are there in a graph with 10 vertices of degree Five?

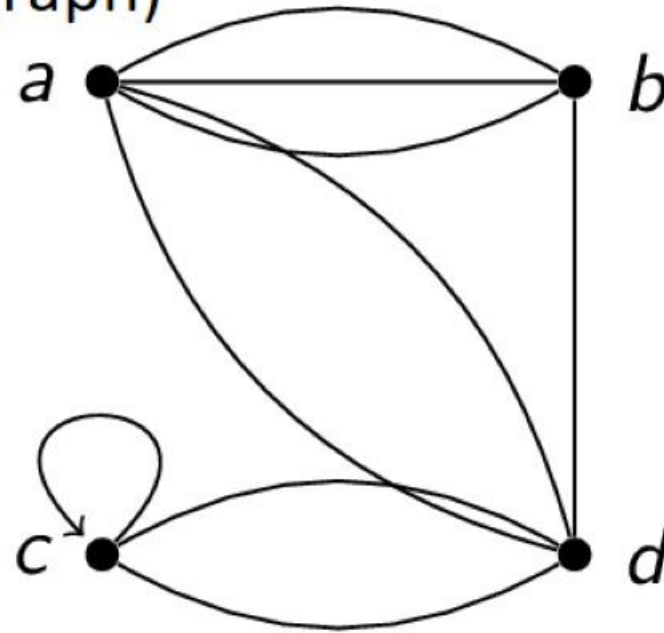
Solution: Because the sum of the degrees of the vertices is $5 \cdot 10 = 50$, the handshaking theorem tells us that $2m = 50$.

So the number of edges $m = 25$.

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

$$\begin{array}{c}
 \\
 a \\
 b \\
 c \\
 d
 \end{array}
 \begin{array}{ccccc}
 & a & b & c & d \\
 \left(\begin{array}{cccc}
 0 & 3 & 0 & 2 \\
 3 & 0 & 0 & 1 \\
 0 & 0 & 1 & 2 \\
 2 & 1 & 2 & 0
 \end{array} \right)
 \end{array}$$

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

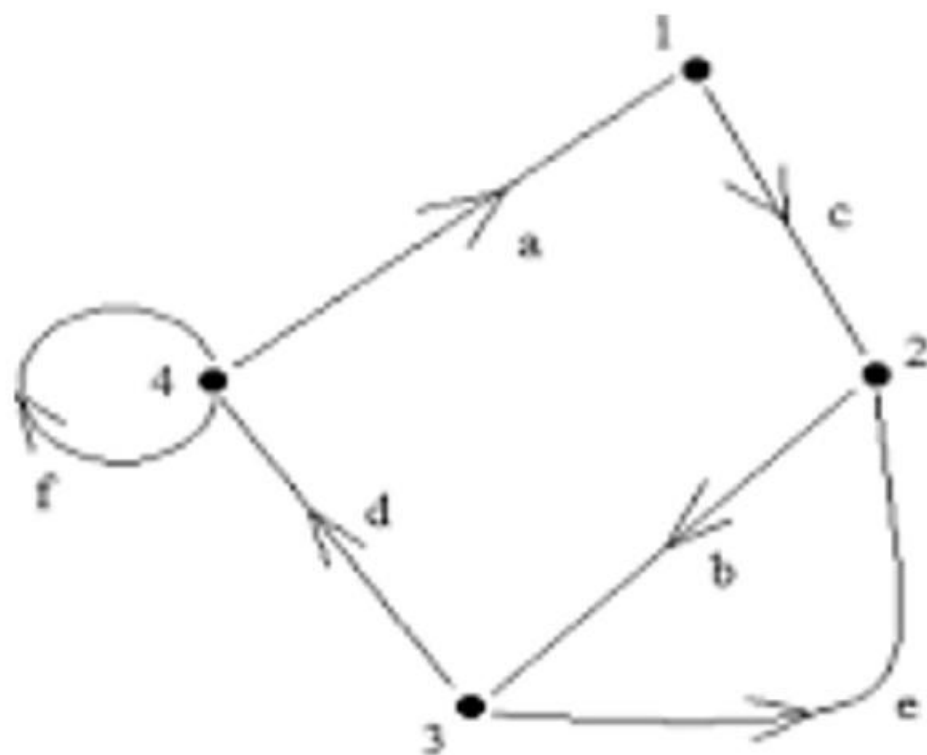


$$\begin{matrix} & a & b & c & d \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \end{matrix}$$

Multigraph loop and undirected

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

$$M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$



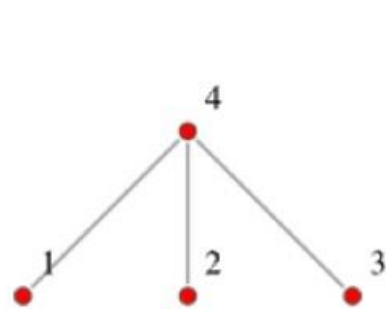
$$M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

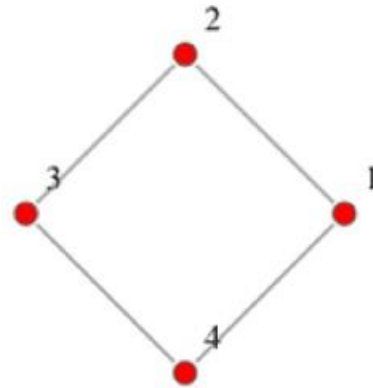
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

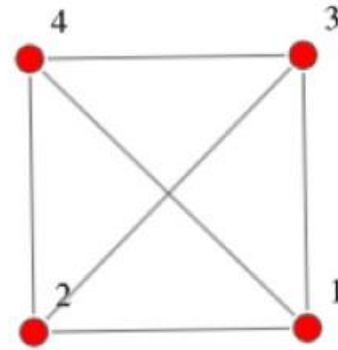
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



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$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Simple graph

Complete

1. There are **no edges** between its vertices.
2. Edge set is **empty**
3. The graph which has only one vertex
4. We use ordered pair to represent ----- graph
5. We use sets to represent edge -----graph
6. A **loop** at a vertex contributes _____ to the degree of that vertex.
7. An *adjacency list* can be used to represent a graph with **multiple edges**?

1. Null graph
2. Null graph
3. Trivial graph
4. Directed
5. Undirected
6. Two
7. False

- Graph
- Types of graph
- Degrees
- Neighborhood
- Adjacency List
- Adjacency Matrix



Thank You