

Mathematics

Lecture 8

MAT 122

Advanced Topics in Linear Algebra

Prepared by
Dr. Israa Lewaa

- Definition of Inverse
- Inverse of 3×3 matrix using Adjoint matrix
- How to check the inverse matrix is achieved or not
- Inverse of 3×3 matrix using Reduced Echelon Form
- Inverse of 3×3 matrix using Scientific Calculator

Inverse of 3×3 matrix using Adjoint matrix
First Method

Find the inverse of the following:

$$A = \begin{bmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix}$$

Solution of Example 1

$$|A| = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix}$$

$$= 9((-1) \times (-2) - 0 \times 6) - 2(5 \times (-2) - 4 \times 6) + 1(5 \times 0 - 4 \times (-1))$$

$$= 9(2 - 0) - 2(-10 - 24) + 1(0 + 4)$$

$$= 9 \times 2 - 2 \times (-34) + 1 \times 4$$

$$= 18 + 68 + 4$$

$$= 90$$

Since $|A| \neq 0$,

Inverse of A is possible

Solution of Example 1

Finding cofactors

$$A_{11} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 6 \\ 0 & -2 \end{vmatrix} = (-1) \times (-2) - 0 \times 6 = 2 - 0 = 2$$

$$A_{12} = -1 \times \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = -1 \times \begin{vmatrix} 5 & 6 \\ 4 & -2 \end{vmatrix} = -(5 \times (-2) - 4 \times 6) = 34$$

$$A_{13} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 5 & -1 \\ 4 & 0 \end{vmatrix} = 5 \times 0 - 4 \times (-1) = 0 + 4 = 4$$

Solution of Example 1

$$A_{21} = -1 \times \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = -1 \times \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} = -(2 \times (-2) - 0 \times 1) = 4$$

$$A_{22} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & 1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 1 \\ 4 & -2 \end{vmatrix} = (9 \times (-2) - 4 \times 1) = -22$$

$$A_{23} = -1 \times \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = -1 \times \begin{vmatrix} 9 & 2 \\ 4 & 0 \end{vmatrix} = -(9 \times 0 - 4 \times 2) = 8$$

$$A_{31} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix} = 2 \times 6 - (-1) \times 1 = 12 + 1 = 13$$

$$A_{32} = -1 \times \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & 2 \end{vmatrix} = -1 \times \begin{vmatrix} 9 & 1 \\ 5 & 6 \end{vmatrix} = -(9 \times 6 - 5 \times 1) = -49$$

$$A_{33} = \begin{vmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 9 & 2 \\ 5 & -1 \end{vmatrix} = 9 \times (-1) - 5 \times 2 = -9 - 10 = -19$$

Thus,

$$\text{adj } A = \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$

And,

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{90} \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$

$$= \frac{1}{90} \begin{bmatrix} 2 & 4 & 13 \\ 34 & -22 & -49 \\ 4 & 8 & -19 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{90} & \frac{4}{90} & \frac{13}{90} \\ \frac{34}{90} & \frac{-22}{90} & \frac{-49}{90} \\ \frac{4}{90} & \frac{8}{90} & \frac{-19}{90} \end{bmatrix} = \begin{bmatrix} \frac{1}{45} & \frac{2}{45} & \frac{13}{90} \\ \frac{17}{45} & \frac{-11}{45} & \frac{-49}{90} \\ \frac{2}{45} & \frac{4}{45} & \frac{-19}{90} \end{bmatrix}$$

$$AA^{-1}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{45} & \frac{2}{45} & \frac{13}{90} \\ \frac{17}{45} & \frac{-11}{45} & \frac{-49}{90} \\ \frac{2}{45} & \frac{4}{45} & \frac{-19}{90} \end{bmatrix}$$

$$= \begin{bmatrix} 9\left(\frac{1}{45}\right) + 2\left(\frac{17}{45}\right) + 1\left(\frac{2}{45}\right) & 9\left(\frac{2}{45}\right) + 2\left(\frac{-11}{45}\right) + 1\left(\frac{4}{45}\right) & 9\left(\frac{13}{90}\right) + 2\left(\frac{-49}{90}\right) + 1\left(\frac{-19}{90}\right) \\ 5\left(\frac{1}{45}\right) + (-1)\left(\frac{17}{45}\right) + 6\left(\frac{2}{45}\right) & 5\left(\frac{2}{45}\right) + (-1)\left(\frac{-11}{45}\right) + 6\left(\frac{4}{45}\right) & 5\left(\frac{13}{90}\right) + (-1)\left(\frac{-49}{90}\right) + 6\left(\frac{-19}{90}\right) \\ 4\left(\frac{1}{45}\right) + 0\left(\frac{17}{45}\right) + (-2)\left(\frac{2}{45}\right) & 4\left(\frac{2}{45}\right) + 0\left(\frac{-11}{45}\right) + (-2)\left(\frac{4}{45}\right) & 9\left(\frac{13}{90}\right) + 0\left(\frac{-49}{90}\right) + (-2)\left(\frac{-19}{90}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{9}{45} + \frac{34}{45} + \frac{2}{45} & \frac{18}{45} - \frac{22}{45} + \frac{4}{45} & \frac{117}{90} - \frac{98}{90} - \frac{19}{90} \\ \frac{5}{45} - \frac{17}{45} + \frac{12}{45} & \frac{10}{45} + \frac{11}{45} + \frac{24}{45} & \frac{65}{90} + \frac{49}{90} - \frac{114}{90} \\ \frac{4}{45} - \frac{4}{45} & \frac{8}{45} - \frac{8}{45} & \frac{52}{90} + \frac{38}{90} \end{bmatrix}$$

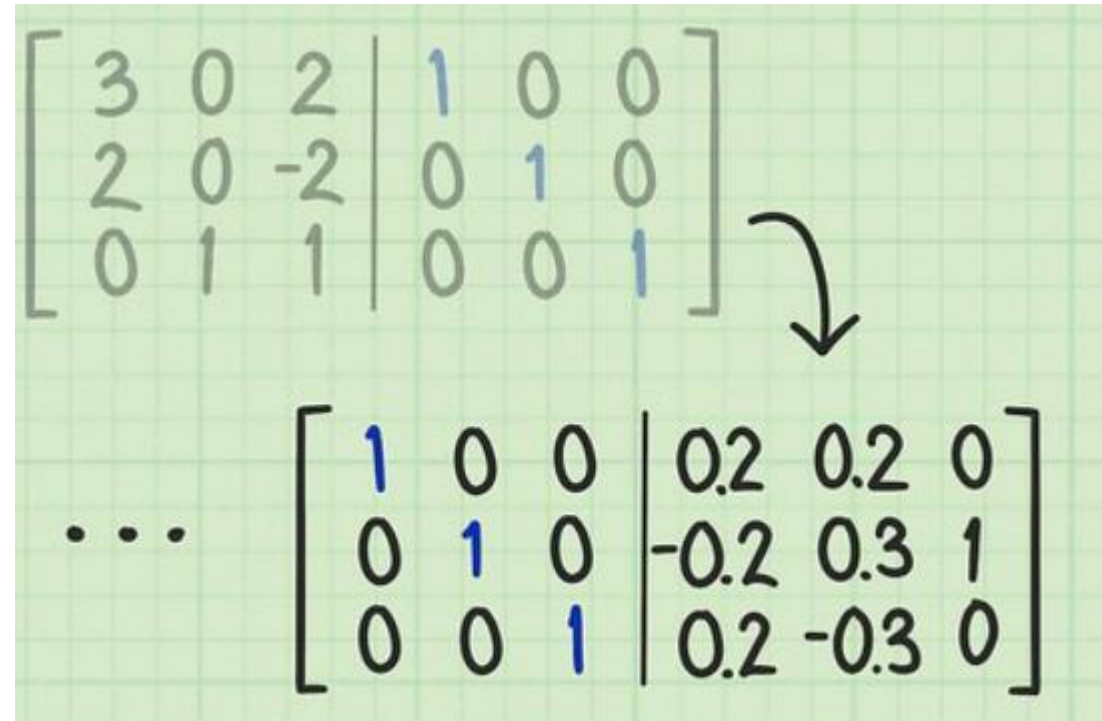
$$= \begin{bmatrix} \frac{45}{45} & 0 & 0 \\ 0 & \frac{45}{45} & 0 \\ 0 & 0 & \frac{90}{90} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

Thus,
 $AA^{-1} = \mathbf{I}$

Inverse of 3×3 matrix using Reduced Echelon Form
Second Method

$$M = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}; \quad M^{-1} = ?$$
$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{bmatrix} 3 & 0 & 2 & | & 1 & 0 & 0 \\ 2 & 0 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \quad R1' = R1 + R2$$
$$\begin{bmatrix} 5 & 0 & 0 & | & 1 & 1 & 0 \\ 2 & 0 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \quad R1' = R1 \div 5$$
$$\begin{bmatrix} 1 & 0 & 0 & | & 0.2 & 0.2 & 0 \\ 2 & 0 & -2 & | & 0 & 1 & 0 \\ 0 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \quad \dots$$

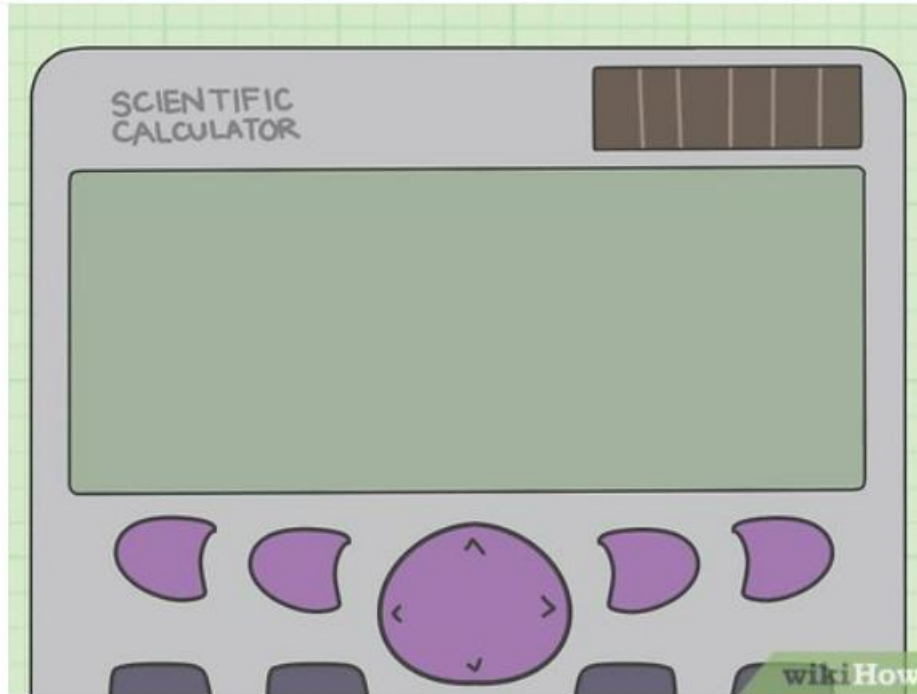


The image shows a handwritten matrix reduction process on green grid paper. The first matrix is a 3x6 augmented matrix with a vertical line separating the coefficients from the identity matrix. The first row is [3, 0, 2 | 1, 0, 0], the second row is [2, 0, -2 | 0, 1, 0], and the third row is [0, 1, 1 | 0, 0, 1]. A curved arrow points from this matrix to a second matrix below it. The second matrix is also a 3x6 augmented matrix, with the first row being [1, 0, 0 | 0.2, 0.2, 0], the second row being [0, 1, 0 | -0.2, 0.3, 1], and the third row being [0, 0, 1 | 0.2, -0.3, 0]. Ellipses (...) are written to the left of the second matrix.

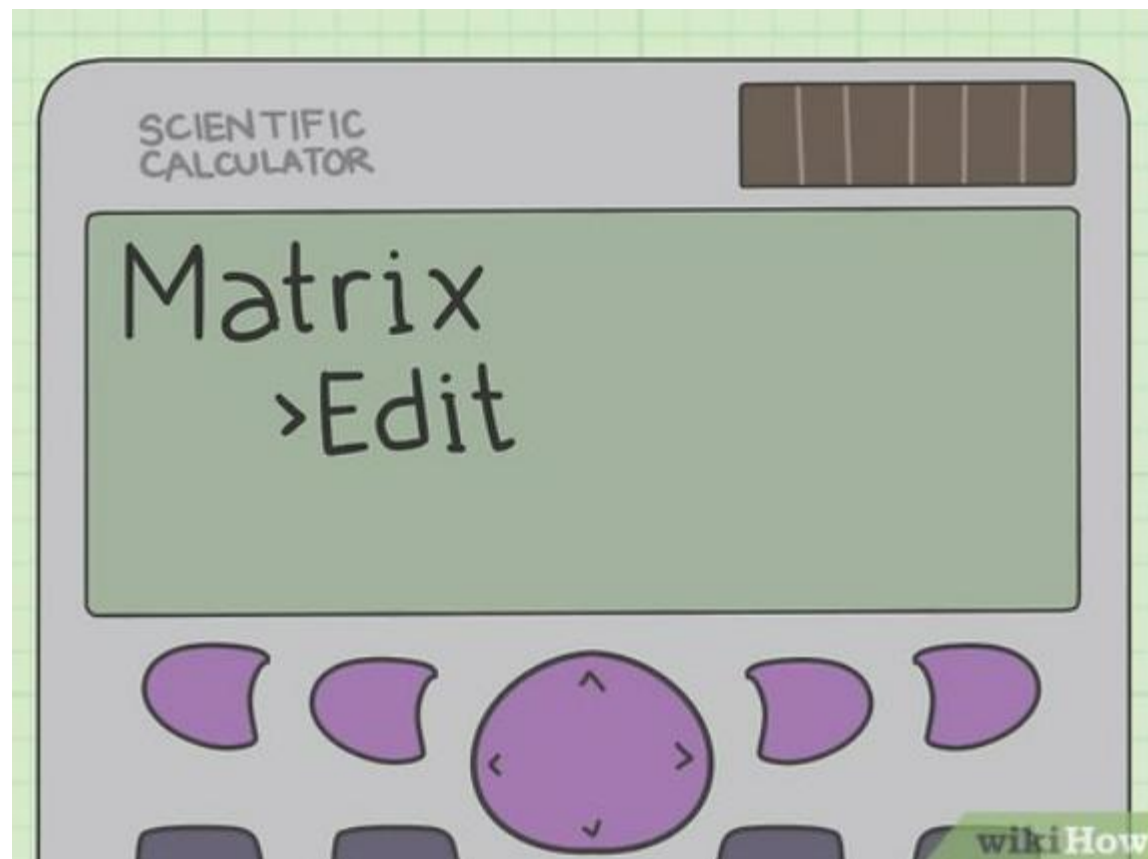
$$\left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \rightarrow \dots \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array} \right]$$

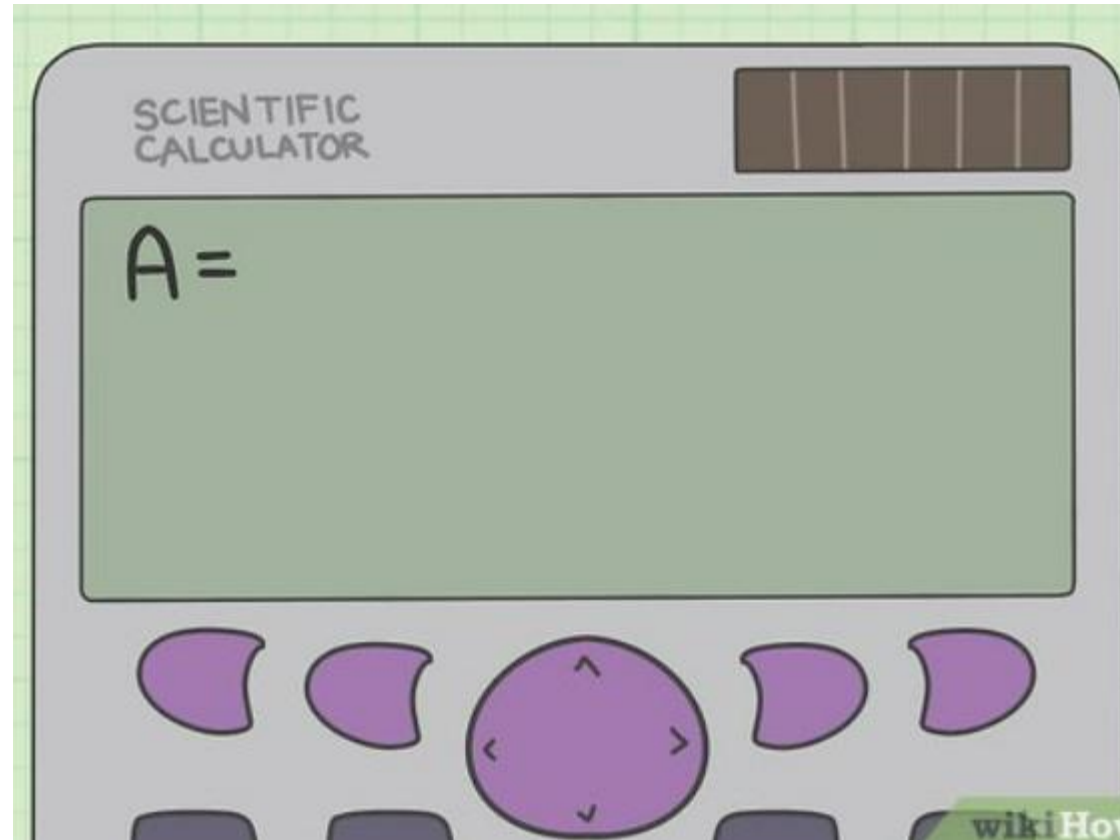
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 1 & 0 & -0.2 & 0.3 & 1 \\ 0 & 0 & 1 & 0.2 & -0.3 & 0 \end{array} \right]$$
$$M^{-1} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

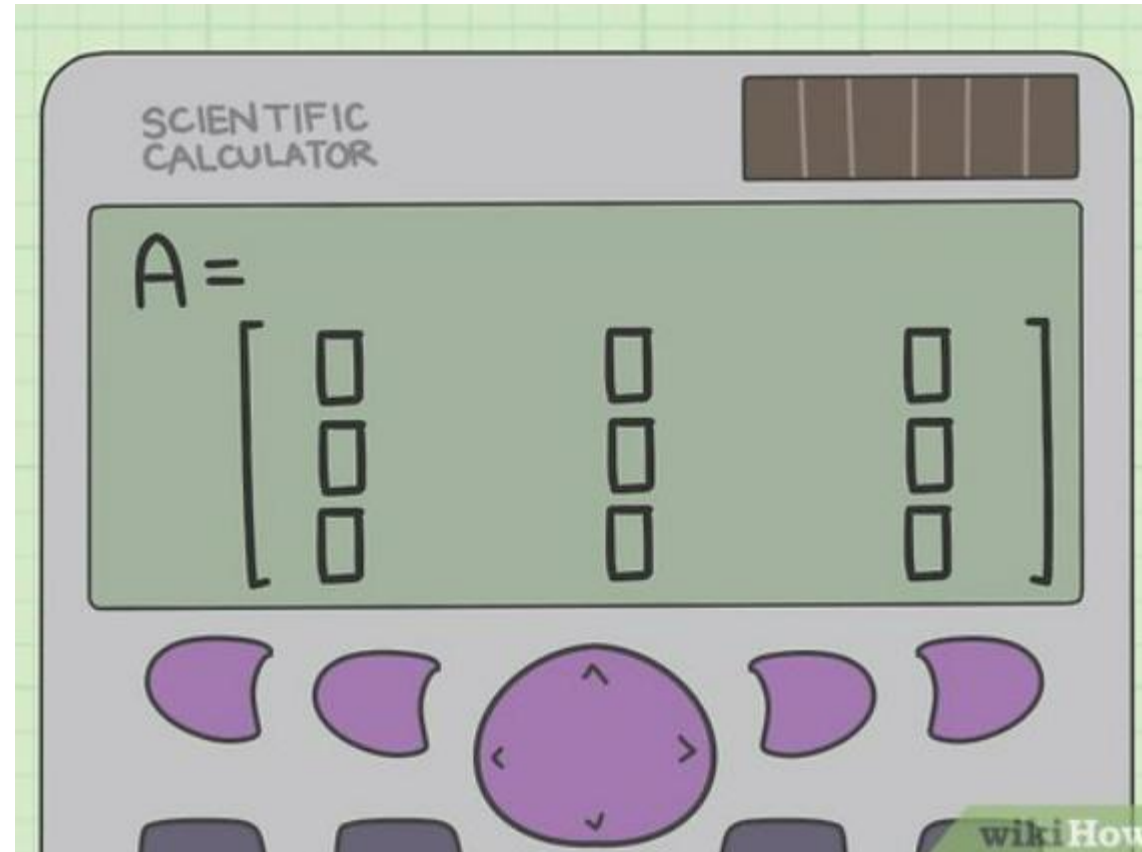
Inverse of 3×3 matrix using Calculator
Third Method

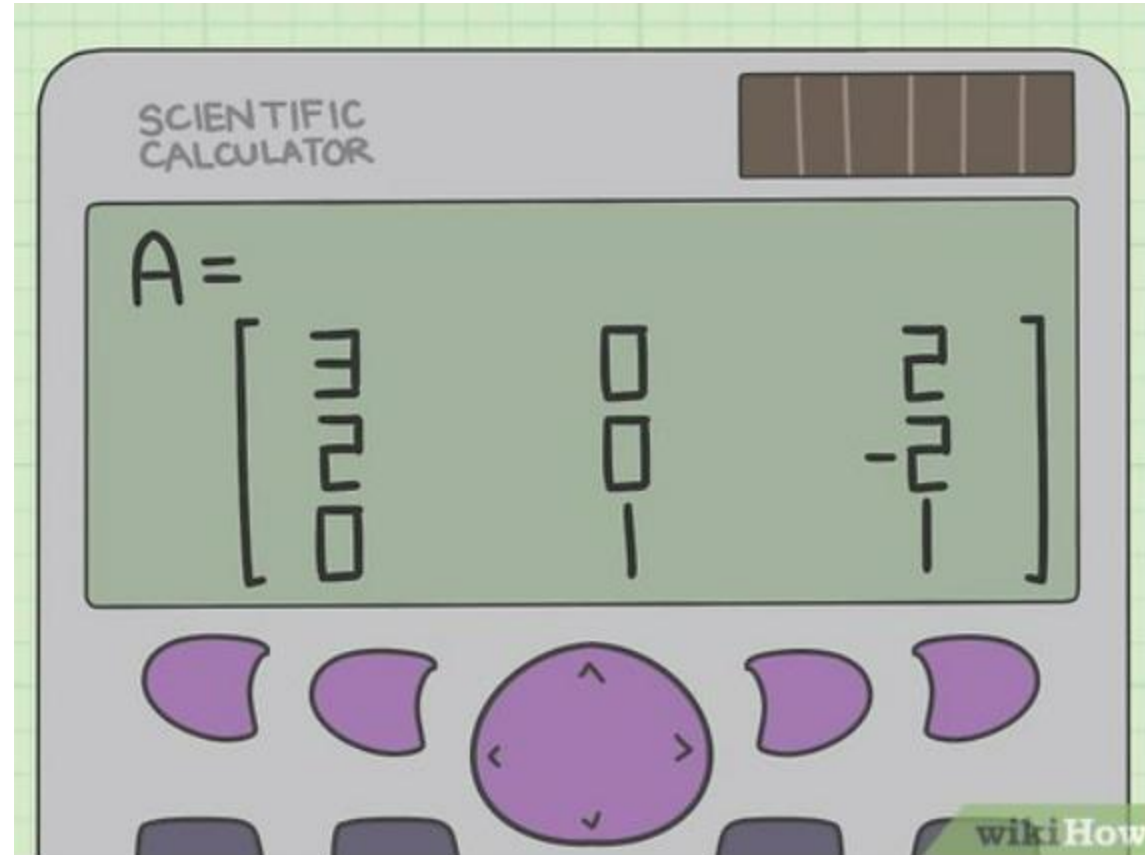


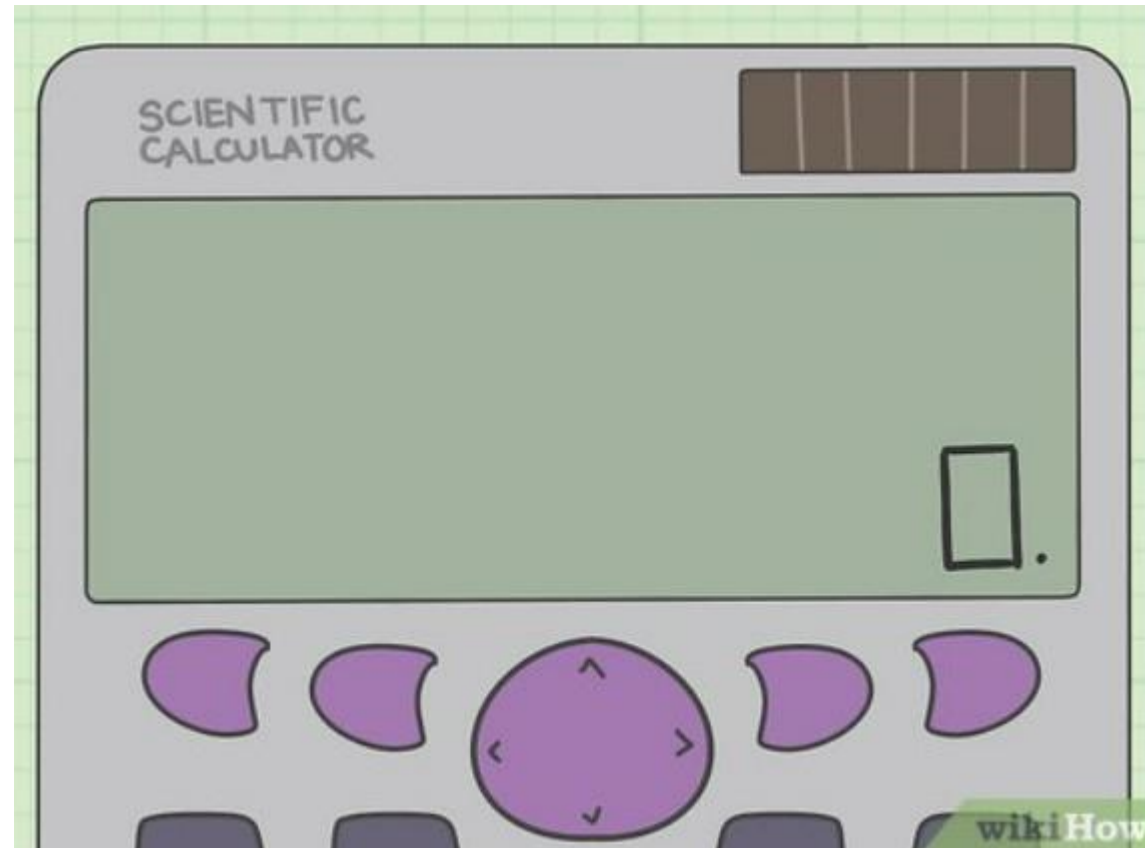
fx 991

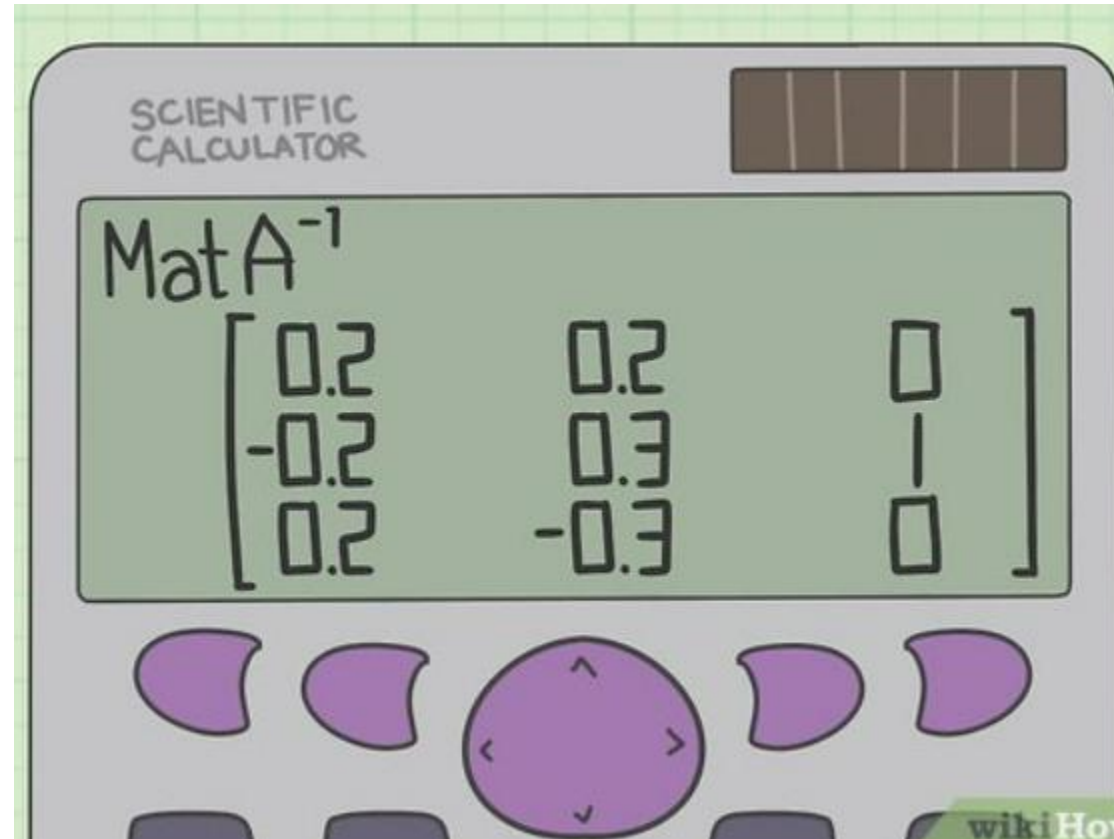


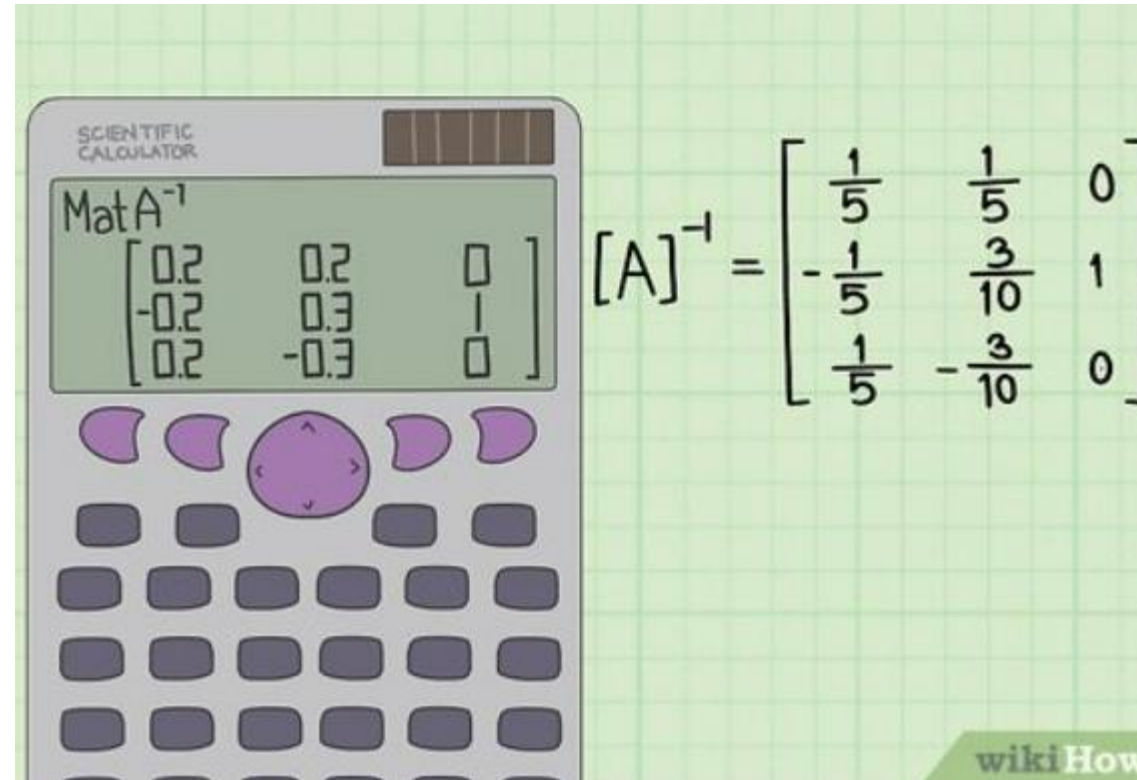












Let's Practice

Find the inverse of the following:

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we have $\det(A) = -1 \neq 0$, and hence A is invertible.

To find the inverse using the formula, we first determine the cofactors C_{ij} of A .

We have

$$C_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \quad C_{12} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{21} = -\begin{vmatrix} 5 & 2 \\ 0 & 1 \end{vmatrix} = -5, \quad C_{22} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{23} = -\begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 5 & 2 \\ -1 & 2 \end{vmatrix} = 12, \quad C_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -2, \quad C_{33} = \begin{vmatrix} 1 & 5 \\ 0 & -1 \end{vmatrix} = -1.$$

The the adjoint matrix of A is

$$\text{Adj}(A) = C^T = \begin{bmatrix} -1 & -5 & 12 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Using the formula, we obtain the inverse matrix

$$A^{-1} = \frac{1}{\det(A)} \text{Adj}(A) = \begin{bmatrix} 1 & 5 & -12 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the inverse of the following:

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}.$$

First, we have to obtain the value of the determinant

$$\det(B) = -5 \neq 0.$$

So the matrix B is invertible.

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}.$$

Now the cofactors C_{ij} of B are

$$C_{11} = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{12} = -\begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} = 12, \quad C_{13} = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5, \quad C_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}.$$

Now the cofactors C_{ij} of B are

$$C_{11} = \begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{12} = -\begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} = 12, \quad C_{13} = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = -3$$

$$C_{21} = -\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5, \quad C_{23} = -\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Hence the adjoint matrix of B is

$$\text{Adj}(B) = C^T = \begin{bmatrix} 1 & 0 & -2 \\ 12 & -5 & -4 \\ -3 & 0 & 1 \end{bmatrix}.$$

It follows from the formula that the inverse matrix of B is

$$B^{-1} = \frac{1}{\det(B)} \text{Adj}(B) = \frac{1}{5} \begin{bmatrix} -1 & 0 & 2 \\ -12 & 5 & 4 \\ 3 & 0 & -1 \end{bmatrix}.$$

Find the inverse of the following:

$$\begin{bmatrix} 1 & 0 & -2 \\ 12 & 0 & -4 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 12 & 0 & -4 \\ -3 & 0 & 1 \end{bmatrix}$$

First, we will obtain the value of the determinant for this matrix:

Det of the matrix = Zero

So, this matrix is noninvertible.

- Definition of Inverse
- Inverse of 3×3 matrix using Adjoint matrix
- How to check the inverse matrix is achieved or not
- Inverse of 3×3 matrix using Reduced Echelon Form
- Inverse of 3×3 matrix using Scientific Calculator