

Mathematics

Lecture 2

MAT 112

Exponential ang Logarithmic Functions

Prepared by
Dr. Israa Lewaa

- To introduce exponential functions and their applications.
 - To introduce logarithmic functions and their graphs.
 - To study the basic properties of logarithmic functions.
 - To develop techniques for solving logarithmic and exponential equations.
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Exponential Functions

Logarithmic Functions

Properties of Logarithms

Logarithmic and Exponential Equations

1. Exponential Functions

The function f defined by $f(x) = b^x$

where $b > 0$, $b \neq 1$, and the exponent x is any real number, is called an ***exponential function*** with base b .

1. Exponential Functions

Rules for Exponents

1. $a^m a^n = a^{m+n}$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

2. $\frac{a^m}{a^n} = a^{m-n}$

6. $a^1 = a$

3. $(a^m)^n = a^{mn}$

7. $a^0 = 1$

4. $(ab)^n = a^n b^n$

8. $a^{-n} = \frac{1}{a^n}$

Example 1 – Bacteria Growth

The number of bacteria present in a culture after t minutes is given by $N(t) = 200\left(\frac{4}{3}\right)^t$.

- a. How many bacteria are present initially?*
 - b. Approximately how many bacteria are present after 3 minutes?*
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Solution:

a. When $t = 0$, $N(0) = 200 \left(\frac{4}{3} \right)^0 = 200 (1) = 200$

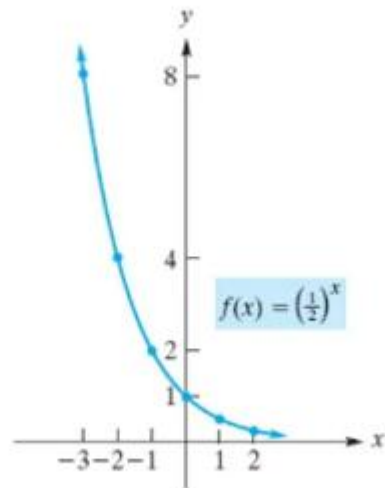
b. When $t = 3$, $N(3) = 200 \left(\frac{4}{3} \right)^3 = 200 \left(\frac{64}{27} \right) = \frac{12800}{27} \approx 474$

Example – Graphing Exponential Functions with $0 < b < 1$

Graph the exponential function $f(x) = (1/2)^x$.

Solution:

x	$\left(\frac{1}{2}\right)^x$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



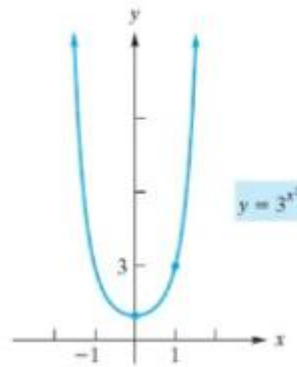
Properties of Exponential Functions

1. The domain of an exponential function consists of all real numbers.
The range consists of all positive numbers.
 2. The graph of $f(x) = b^x$ has y-intercept $(0, 1)$.
There is no x -intercept.
 3. If $b > 1$, the graph *rises* from left to right.
If $0 < b < 1$, the graph *falls* from left to right.
 4. If $b > 1$, the graph approaches the x -axis as x becomes more and more negative.
If $0 < b < 1$, the graph approaches the x -axis as x becomes more and more positive.
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Example – Graph of a Function with a Constant Base

Graph $y = 3^{x^2}$.

Solution:



x	0	1	2
y	1	3	81

Compound Interest

- The compound amount S of the principal P at the end of n years at the rate of r compounded annually is given by

$$S = P(1+r)^n$$

Example – Population Growth

The population of a town of 10,000 grows at the rate of 2% per year. Find the population three years from now.

Solution:

For $t = 3$, we have $P(3) = 10,000(1.02)^3 \approx 10,612$.

Example – Population Growth

The projected population P of a city is given by $P = 100,000e^{0.05t}$ where t is the number of years after 1990. Predict the population for the year 2010.

Solution:

For $t = 20$,

$$P = 100,000e^{0.05(20)} = 100,000e^1 = 100,000e \approx 271,828$$

Example – Radioactive Decay

A radioactive element decays such that after t days the number of milligrams present is given by

$$N = 100e^{-0.062t}.$$

- a. How many milligrams are initially present?*
 - b. How many milligrams are present after 10 days?*
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a. How many milligrams are initially present?

Solution: For $t = 0$, $N = 100e^{-0.062(0)} = 100$ mg.

b. How many milligrams are present after 10 days?

Solution: For $t = 10$, $N = 100e^{-0.062(10)} = 53.8$ mg

Logarithmic Functions

- $y = \log_b x$ if and only if $b^y = x$.
- Fundamental equations are $\log_b b^x = x$ and $b^{\log_b x} = x$

Example – Converting from Exponential to Logarithmic Form

Exponential Form

Logarithmic Form

a. Since $5^2 = 25$

it follows that $\log_5 25 = 2$

b. Since $3^4 = 81$

it follows that $\log_3 81 = 4$

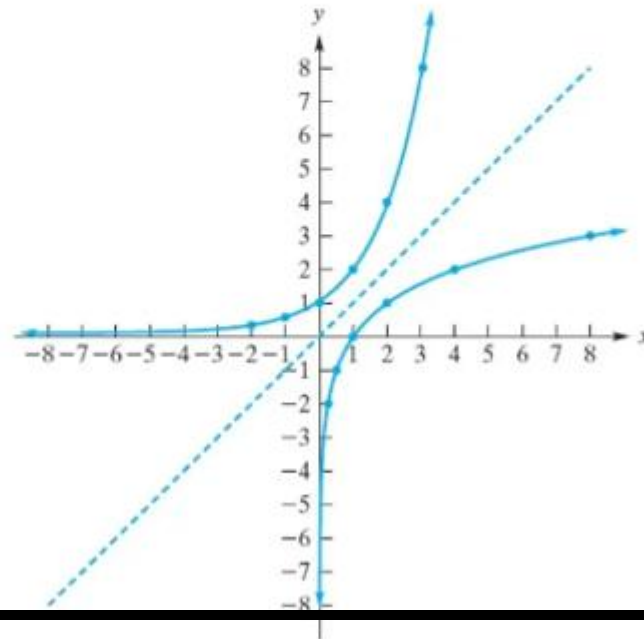
c. Since $10^0 = 1$

it follows that $\log_{10} 1 = 0$

Example – Graph of a Logarithmic Function with $b > 1$

Sketch the graph of $y = \log_2 x$.

Solution:



Properties of Logarithms

- Properties of logarithms are:

1. $\log_b(mn) = \log_b m + \log_b n$

2. $\log_b \frac{m}{n} = \log_b m - \log_b n$

3. $\log_b m^r = r \log_b m$

4. $\log_b \frac{1}{m} = -\log_b m$

5. $\log_b 1 = 0$

6. $\log_b b = 1$

7. $\log_b m = \frac{\log_a m}{\log_a b}$

Example 1 – Finding Logarithms

a. $\log 56 = \log(8 \cdot 7) = \log 8 + \log 7 \approx 0.9031 + 0.8451 = 1.7482$

b. $\log \frac{9}{2} = \log 9 - \log 2 \approx 0.9542 - 0.3010 = 0.6532$

c. $\log 64 = \log 8^2 = 2\log 8 \approx 2(0.9031) = 1.8062$

d. $\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2}\log 5 \approx \frac{1}{2}(0.6990) = 0.3495$

Example – Finding Logarithms

a. Find $\log 100$.

$$\log 100 = \log(10)^2 = 2$$

b. Find $\ln 1$.

$$\ln 1 = 0$$

c. Find $\log 0.1$.

$$\log 0.1 = \log 10^{-1} = -1$$

d. Find $\ln e^{-1}$.

$$\ln e^{-1} = -1 \ln e = -1$$

d. Find $\log_{36} 6$.

$$\log_{36} 6 = \frac{\log 6}{2 \log 6} = \frac{1}{2}$$

Example – Writing Logarithms in Terms of Simpler Logarithms

a. $\ln \frac{x}{zw} = \ln x - \ln(zw)$

$$= \ln x - (\ln z + \ln w)$$

$$= \ln x - \ln z - \ln w$$

b. $\ln \sqrt[3]{\frac{x^5(x-2)^8}{x-3}} = \ln \left[\frac{x^5(x-2)^8}{x-3} \right]^{1/3} = \frac{1}{3} \ln \frac{x^5(x-2)^8}{x-3}$

$$= \frac{1}{3} \{ \ln [x^5(x-2)^8] - \ln(x-3) \}$$

$$= \frac{1}{3} [\ln x^5 + \ln(x-2)^8 - \ln(x-3)]$$

$$= \frac{1}{3} [5 \ln x + 8 \ln(x-2) - \ln(x-3)]$$

Example – Simplifying Logarithmic Expressions

a. $\ln e^{3x} = 3x.$

b. $\log 1 + \log 1000 = 0 + \log 10^3$
 $= 0 + 3$
 $= 3$

c. $\log_7 \sqrt[9]{7^8} = \log_7 7^{8/9} = \frac{8}{9}$

d. $\log_3 \left(\frac{27}{81} \right) = \log_3 \left(\frac{3^3}{3^4} \right) = \log_3 (3^{-1}) = -1$

e. $\ln e + \log \frac{1}{10} = \ln e + \log 10^{-1}$
 $= 1 + (-1) = 0$

Example – Evaluating a Logarithm Base 5

Find $\log_5 2$.

Solution: $5^x = 2$
 $\log 5^x = \log 2$
 $x \log 5 = \log 2$
 $x = \frac{\log 2}{\log 5} \approx 0.4307$

Logarithmic and Exponential Equations

- A **logarithmic equation** involves the logarithm of an expression containing an unknown.
 - An **exponential equation** has the unknown appearing in an exponent.
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Example 1 – Oxygen Composition

An experiment was conducted with a particular type of small animal. The logarithm of the amount of oxygen consumed per hour was determined for a number of the animals and was plotted against the logarithms of the weights of the animals. It was found that

$$\log y = \log 5.934 + 0.885 \log x$$

where y is the number of microliters of oxygen consumed per hour and x is the weight of the animal (in grams). Solve for y .

Solution:

$$\log y = \log 5.934 + 0.885 \log x$$

$$= \log 5.934 + \log x^{0.885}$$

$$\log y = \log(5.934 x^{0.885})$$

$$y = 5.934 x^{0.885}$$

Example – Using Logarithms to Solve an Exponential Equation

Solve $5 + (3)4^{x-1} = 12$.

Solution:

$$5 + (3)4^{x-1} = 12$$
$$4^{x-1} = \frac{7}{3}$$
$$\ln 4^{x-1} = \ln \frac{7}{3}$$
$$x \approx 1.61120$$

Find y

$$\text{Log } (y) = 2 \log (5) + 3 \log (x)$$

$$\text{Log } (y) = \log (5)^2 + \log (x)^3$$

$$\text{Log } (y) = \log (25x^3)$$

$$\therefore Y = 25 x^3$$

Exponential Functions

How to graph Exponential Functions

Logarithmic Functions

Properties of Logarithms

Logarithmic and Exponential Equations
