

Business Mathematics

Prepared by Dr. Faisal Hasan

Lecture NO 4

جامعة الجميع الذكية

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MTH 112

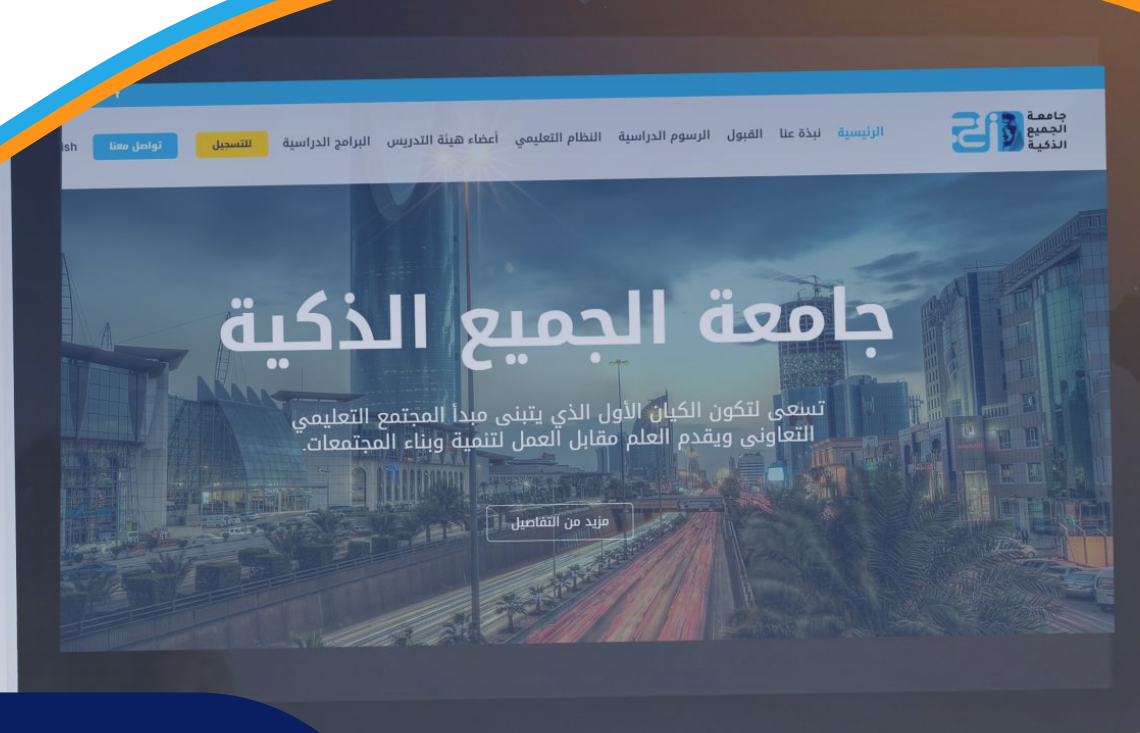
Business Mathematics



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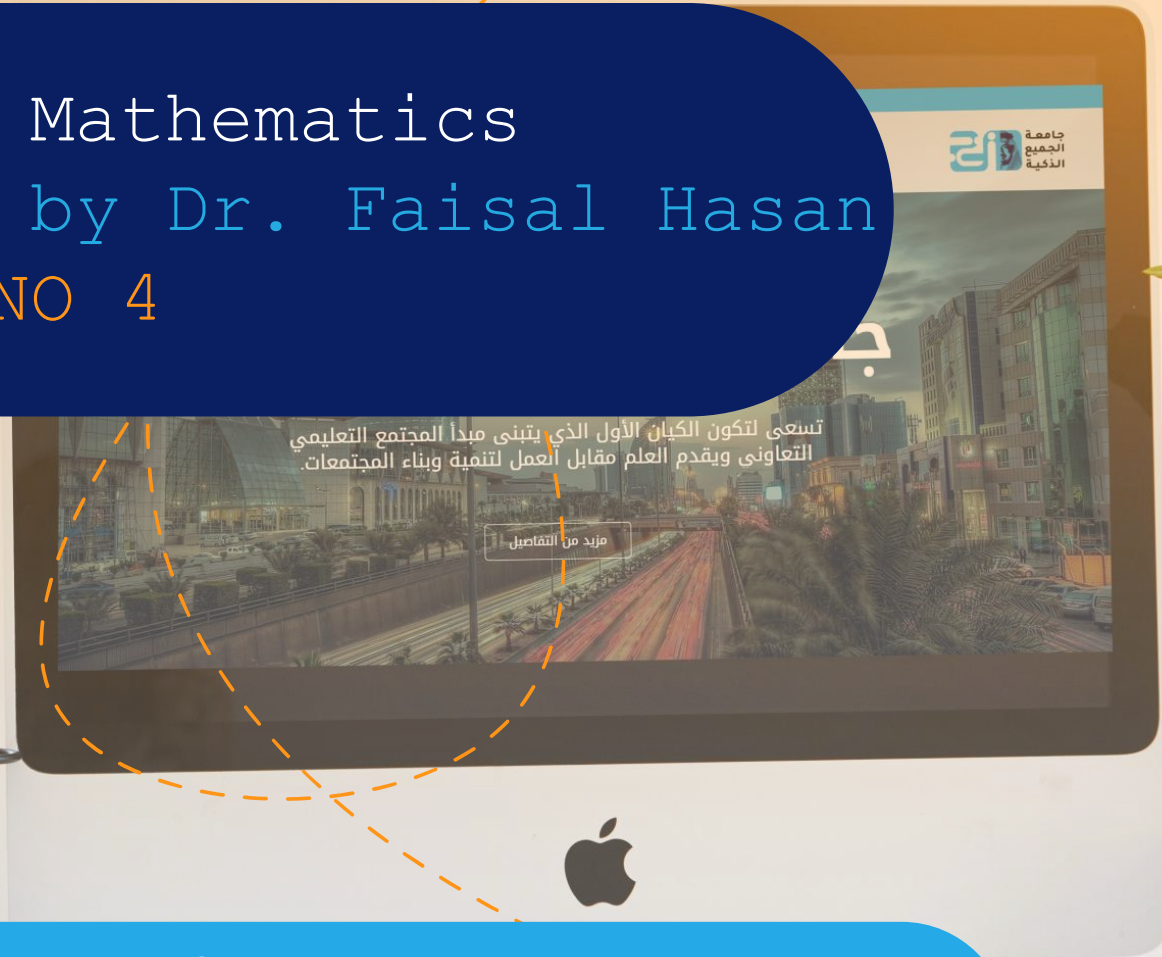
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Business Mathematics

Topics

- Chapter 1: Mathematical preliminaries
- Chapter 2: The straight line and applications
- Chapter 3: Functions and Graphs
- Chapter 4: Limits and Continuity
- Chapter 5: Financial Mathematics
- Chapter 6: Differentiation
- Chapter 7: Methods of Differentiation
- Chapter 8: Function of several variables
- Chapter 9: Integration and applications
- Chapter 10: Linear algebra and applications

- 1: The Concept of Limits
- 2: Computation of Limits
- 3: Methods of Limits
- 4: Continuity



- When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it as little as one wishes, this last is called the limit of all the others.

$$\lim_{x \rightarrow a} f(x) = L$$

$$\lim_{x \rightarrow 2} x^2 + 1 = (2)^2 + 1 = 5$$

Positive value

$$\lim_{x \rightarrow 0} \frac{3}{x + 4} = \frac{3}{0 + 4} = \frac{3}{4}$$

$$\lim_{x \rightarrow -4} 2x + 1 = 2(-4) + 1 = -7$$

Negative value

$$\lim_{x \rightarrow -3} \frac{x}{x + 10} = \frac{-3}{-3 + 10} = -\frac{3}{7}$$

$$\lim_{x \rightarrow 2} x^2 - 4 = (2)^2 - 4 = 0$$

Zero

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x + 4} = \frac{(1)^3 - 1}{1 + 4} = \frac{0}{5} = 0$$

- Divided any number by infinity ∞ , we will obtain Zero
- Divided any number by Zero 0 , we will obtain infinity ∞

$$\frac{2}{\infty} = 0 , \quad \frac{4}{0} = \infty$$

$$\lim_{x \rightarrow 3} \frac{4}{x - 3} = \frac{4}{3 - 3} = \frac{4}{0} = \infty$$

Infinity

$$\lim_{x \rightarrow 4} \frac{x^3 - 1}{x - 4} = \frac{(4)^3 - 1}{4 - 4} = \frac{64 - 1}{0} = \frac{63}{0} = \infty$$

$$\lim_{x \rightarrow \infty} \frac{4}{x + 3} = \frac{4}{\infty + 3} = \frac{4}{\infty} = 0$$

Zero

$$\lim_{x \rightarrow \infty} \frac{1}{2x + 12} = \frac{1}{\infty + 12} = \frac{1}{\infty} = 0$$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0} \quad ???$$

- If the first substitution give **Undefined Value**

$$\frac{\infty}{\infty}, \frac{0}{0}, \infty - \infty, 1^{\infty}, 0^0$$

We Should solve the Limits

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0}$$

To get the limit of function $f(x)$ as x approach to 2

Chapter 04

The Concept of Limits

x	$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.99999	3.99999

$\simeq 4$

x	$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.00001	4.00001

$\simeq 4$

We can say that

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)}$$

Factorization

$$\lim_{x \rightarrow 2} \frac{\cancel{(x - 2)}(x + 2)}{\cancel{(x - 2)}} = \lim_{x \rightarrow 2} (x + 2) = 4$$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \rightarrow a} f(x) \quad \text{and} \quad \lim_{x \rightarrow a} g(x)$$

exist. Then

$$1. \lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$2. \lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$3. \lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

$$4. \lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \quad \text{if } \lim_{x \rightarrow a} g(x) \neq 0$$



- These five laws can be stated verbally as follows:

- **Sum Law**

1. The limit of a sum is the sum of the limits.

- **Difference Law**

2. The limit of a difference is the difference of the limits.

- **Constant Multiple Law** 3. The limit of a constant times a function is the constant times the limit of the function.
- **Product Law** 4. The limit of a product is the product of the limits.
- **Quotient Law** 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$, provided the limits exist.

$$\lim_{x \rightarrow 2} (3x^2 - 5x + 4)$$

$$\lim_{x \rightarrow 2} (3x^2 - 5x + 4) = 3(2)^2 - 5(2) + 4$$

$$12 - 10 + 4 = 6$$

$$\lim_{x \rightarrow 3} \frac{(x^3 - 5x + 4)}{x^2 - 2}$$

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{(x^3 - 5x + 4)}{x^2 - 2} &= \frac{(3)^3 - 5(3) + 4}{(3)^2 - 2} \\ &= \frac{16}{7} \end{aligned}$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0}$$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)}$$

Factorization

$$\lim_{x \rightarrow 1} \frac{\cancel{(x - 1)}(x + 1)}{\cancel{(x - 1)}} = \lim_{x \rightarrow 1} (x + 1) = 2$$

$$\lim_{x \rightarrow 2} \sqrt[5]{3x^2 - 2x}$$

$$\lim_{x \rightarrow 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{3(2)^2 - 2(2)}$$

$$= \sqrt[5]{8}$$



$$\lim_{x \rightarrow 0} \frac{1 - e^{2x}}{e^x}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - e^{2x}}{e^x} &= \frac{1 - e^{2(0)}}{e^0} = \frac{1 - 1}{1} \\ &= 0 \end{aligned}$$

Given that function

$$f(x) = \begin{cases} 2x & x < 2 \\ x^2 & x \geq 2 \end{cases}$$

Find

$$\lim_{x \rightarrow 2} f(x)$$

One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$x < 2 \quad \Rightarrow \quad \lim_{x \rightarrow 2^-} 2x$$

$$\lim_{x \rightarrow 2^+} x^2 \quad \Leftarrow \quad x > 2$$

$$\lim_{x \rightarrow 2^-} 2x = 2(2) = 4$$

$$\lim_{x \rightarrow 2^+} x^2 = 2^2 = 4$$

If we have

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = L$$

The limit is exist and we can say

$$\lim_{x \rightarrow 2} f(x) = L$$

If we have

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

The limit is not exist and we can say

$$\lim_{x \rightarrow 2} f(x) \text{ is not exist}$$

One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$x < 2 \quad \Rightarrow \quad \lim_{x \rightarrow 2^-} 2x$$

$$\lim_{x \rightarrow 2^+} x^2 \quad \Leftarrow \quad x > 2$$

$$\lim_{x \rightarrow 2^-} 2x = 2(2) = 4$$

$$\lim_{x \rightarrow 2^+} x^2 = 2^2 = 4$$

we have

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 4$$

The limit is exist and we can say

$$\lim_{x \rightarrow 2} f(x) = 4$$

Given that function

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

Find $\lim_{x \rightarrow 0} f(x)$ $\lim_{x \rightarrow -4} f(x)$ $\lim_{x \rightarrow -1} f(x)$

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

$$0 > -1$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} 3x + 1$$

$$\lim_{x \rightarrow 0} 3x + 1 = 3(0) + 1 = 1$$

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

$$-4 < -1$$

$$\lim_{x \rightarrow -4} f(x) = \lim_{x \rightarrow -4} x^2 + 1$$

$$\lim_{x \rightarrow -4} x^2 + 1 = (-4)^2 + 1 = 17$$

Given that function

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

$$\lim_{x \rightarrow -1} f(x)$$

One-Sided Limits

$$\lim_{x \rightarrow -1^-} f(x)$$

$$\lim_{x \rightarrow -1^+} f(x)$$

$$x < -1 \quad \Rightarrow \quad \lim_{x \rightarrow -1^-} x^2 + 1$$

$$\lim_{x \rightarrow -1^+} 3x + 1 \quad \Leftarrow \quad x > -1$$

$$\lim_{x \rightarrow -1^-} (-1)^2 + 1 = 2$$

$$\lim_{x \rightarrow -1^+} 3x + 1 = 3(-1) + 1 = -2$$

$$\lim_{x \rightarrow -1} f(x) \quad \text{is not exist}$$

Given that function

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

Find $\lim_{x \rightarrow 0} f(x)$ $\lim_{x \rightarrow 5} f(x)$ $\lim_{x \rightarrow 3} f(x)$

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$0 < 3$$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} 2x + 1$$

$$\lim_{x \rightarrow 0} 2x + 1 = 2(0) + 1 = 1$$

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$5 > 3$$

$$\lim_{x \rightarrow 5} f(x) = \lim_{x \rightarrow 5} x^2 - 2$$

$$\lim_{x \rightarrow 5} x^2 - 2 = (5)^2 - 2 = 23$$

Given that function

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$\lim_{x \rightarrow 3} f(x)$$

One-Sided Limits

$$\lim_{x \rightarrow 3^-} f(x)$$

$$\lim_{x \rightarrow 3^+} f(x)$$

$$x < 3 \quad \Rightarrow \quad \lim_{x \rightarrow 3^-} 2x + 1$$

$$\lim_{x \rightarrow 3^+} x^2 - 2 \quad \Leftarrow \quad x > 3$$

$$\lim_{x \rightarrow 3^-} 2x + 1 = 2(3) + 1 = 7$$

$$\lim_{x \rightarrow 3^+} x^2 - 2 = (3)^2 - 2 = 7$$

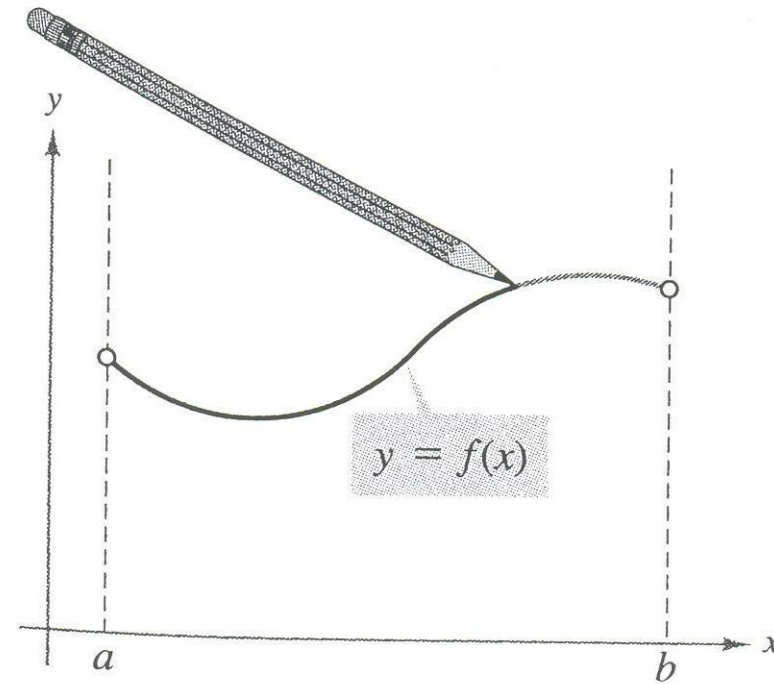
$$\lim_{x \rightarrow 3} f(x) = 7 \text{ is exist}$$

- Continuous functions are predictable...

No breaks in the graph

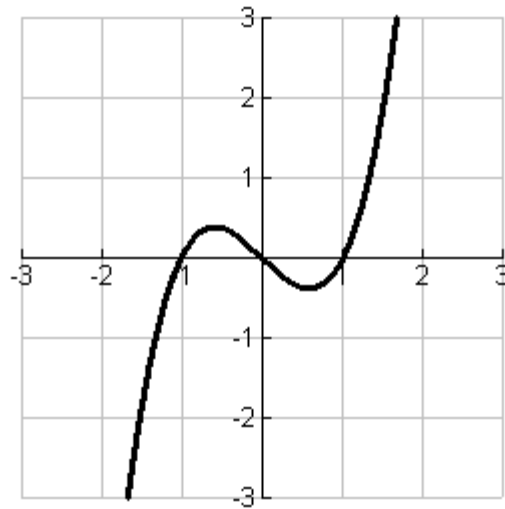
A limit must exist at every x -value or the graph will break.

- Key Point:
Continuous functions can be drawn with a single, unbroken pencil stroke.

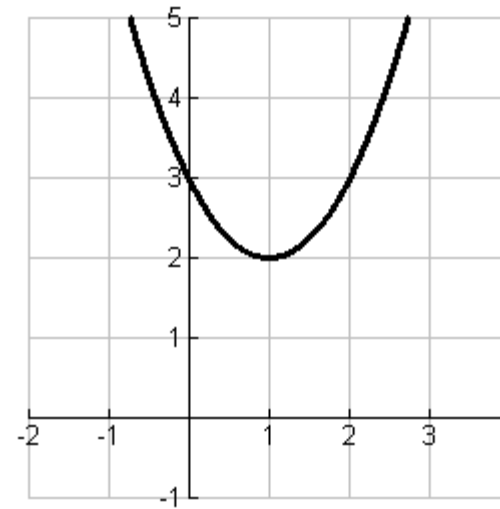


- Mathematically speaking...
If $f(x)$ is continuous, then for every $x = c$ in the function,
- In other words, if you can evaluate any limit on the function using only the substitution method, then the function is continuous.

$$f(x) = x^3 - x$$

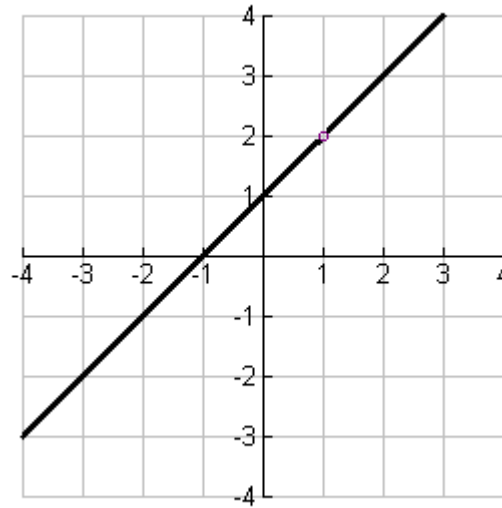


$$f(x) = x^2 - 2x + 3$$



- Both functions are continuous on $(-\infty, \infty)$

$$f(x) = \frac{x^2 - 1}{x - 1}$$



continuous on: $(-\infty, 1) \cup (1, \infty)$

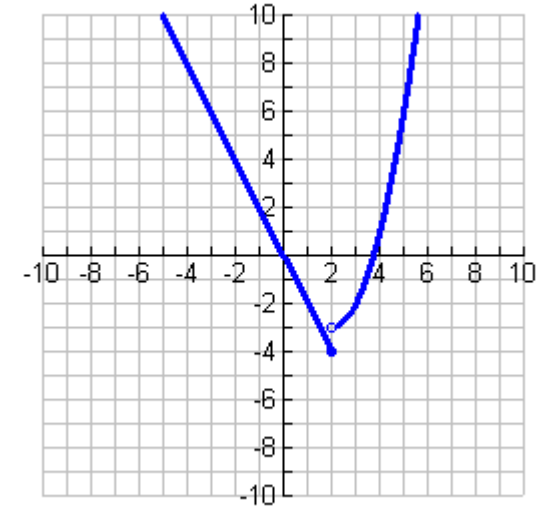
We can say that it continuous on $\mathbb{R}$ except $\{1\}$

$$f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} (-2x) = -4$$

$$\lim_{x \rightarrow 2^+} (x^2 - 4x + 1) = -3$$

$$f(2) = -4$$



Continuous on $\mathbb{R}$ except 2
Function is discontinuous at 2

Discuss the continuity of function at $x = 3$

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$f(3) \text{ ???}$$

$$\lim_{x \rightarrow 3^-} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 3^+} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 3} f(x) \text{ ??? is exist or not}$$

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$f(x) = x^2 - 2$$

$$x = 3$$

$$f(3) = (3)^2 - 2 = 7$$



One-Sided Limits

$$\lim_{x \rightarrow 3^-} f(x)$$

$$\lim_{x \rightarrow 3^+} f(x)$$

$$x < 3 \quad \Rightarrow \quad \lim_{x \rightarrow 3^-} 2x + 1$$

$$\lim_{x \rightarrow 3^+} x^2 - 2 \quad \Leftarrow \quad x > 3$$

$$\lim_{x \rightarrow 3^-} 2x + 1 = 2(3) + 1 = 7$$

$$\lim_{x \rightarrow 3^+} x^2 - 2 = (3)^2 - 2 = 7$$

$$\lim_{x \rightarrow 3} f(x) = 7 \text{ is exist}$$

$$f(x) = \begin{cases} 2x + 1 & x < 3 \\ x^2 - 2 & x \geq 3 \end{cases}$$

$$f(3) = \lim_{x \rightarrow 3} f(x) = 7$$

We can say that it continuous at $x = 3$

Discuss the continuity of function at $x = -1$

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

$$f(-1) \text{ ???}$$

$$\lim_{x \rightarrow -1^-} f(x) \text{ ???}$$

$$\lim_{x \rightarrow -1^+} f(x) \text{ ???}$$

$$\lim_{x \rightarrow -1} f(x) \text{ ??? is exist or not}$$

$$f(x) = \begin{cases} x^2 + 1 & x < -1 \\ 3x + 1 & x \geq -1 \end{cases}$$

$$f(x) = 3x + 1$$

$$f(-1) = 3(-1) + 1 = -2 \quad x = -1$$

One-Sided Limits

$$\lim_{x \rightarrow -1^-} f(x)$$

$$\lim_{x \rightarrow -1^+} f(x)$$

$$x < -1 \quad \Rightarrow \quad \lim_{x \rightarrow -1^-} x^2 + 1$$

$$\lim_{x \rightarrow -1^+} 3x + 1 \quad \Leftarrow \quad x > -1$$

$$\lim_{x \rightarrow -1^-} (-1)^2 + 1 = 2$$

$$\lim_{x \rightarrow -1^+} 3x + 1 = 3(-1) + 1 = -2$$

$$\lim_{x \rightarrow -1} f(x) \quad \text{is not exist}$$

$$f(-1) = -2$$

$$\lim_{x \rightarrow -1} f(x) \text{ is not exist}$$

We can say that it discontinuous at $x = -1$

Discuss the continuity of function at $x = 2$

$$f(x) = \begin{cases} 2x + 1 & x < 2 \\ 5 & x = 2 \\ 3x - 1 & x > 2 \end{cases}$$

$$f(2) \text{ ???}$$

$$\lim_{x \rightarrow 2^-} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 2^+} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 2} f(x) \text{ ??? is exist or not}$$

$$f(x) = \begin{cases} 2x + 1 & x < 2 \\ 5 & x = 2 \\ 3x - 1 & x > 2 \end{cases}$$

$$f(x) = 5 \qquad x = 2$$

One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$x < 2 \quad \rightarrow \quad \lim_{x \rightarrow 2^-} 2x + 1$$

$$\lim_{x \rightarrow 2^+} 3x - 1 \quad \leftarrow \quad x > 2$$

$$\lim_{x \rightarrow 2^-} 2x + 1 = 2(2) + 1 = 5$$

$$\lim_{x \rightarrow 2^+} 3x - 1 = 3(2) - 1 = 5$$

$$\lim_{x \rightarrow 2} f(x) = 5 \text{ is exist}$$

$$f(x) = \begin{cases} 2x + 1 & x < 2 \\ 5 & x = 2 \\ 3x - 1 & x > 2 \end{cases}$$

$$f(2) = \lim_{x \rightarrow 2} f(x) = 5$$

We can say that it continuous at $x = 2$

Discuss the continuity of function at $x = 2$

$$f(x) = \begin{cases} 2x + 3 & x < 2 \\ 8 & x = 2 \\ 3x + 1 & x > 2 \end{cases}$$

$$f(2) \text{ ???}$$

$$\lim_{x \rightarrow 2^-} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 2^+} f(x) \text{ ???}$$

$$\lim_{x \rightarrow 2} f(x) \text{ ??? is exist or not}$$

$$f(x) = \begin{cases} 2x + 3 & x < 2 \\ 8 & x = 2 \\ 3x + 1 & x > 2 \end{cases}$$

$$f(x) = 8 \qquad x = 2$$

One-Sided Limits

$$\lim_{x \rightarrow 2^-} f(x)$$

$$\lim_{x \rightarrow 2^+} f(x)$$

$$x < 2 \quad \Rightarrow \quad \lim_{x \rightarrow 2^-} 2x + 3$$

$$\lim_{x \rightarrow 2^+} 3x + 1 \quad \Leftarrow \quad x > 2$$

$$\lim_{x \rightarrow 2^-} 2x + 3 = 2(2) + 3 = 7$$

$$\lim_{x \rightarrow 2^+} 3x + 1 = 3(2) + 1 = 7$$

$$\lim_{x \rightarrow 2} f(x) = 7 \text{ is exist}$$

$$f(x) = \begin{cases} 2x + 3 & x < 2 \\ 8 & x = 2 \\ 3x + 1 & x > 2 \end{cases}$$
$$f(2) = 8$$
$$\lim_{x \rightarrow 2} f(x) = 7$$
$$f(2) \neq \lim_{x \rightarrow 2} f(x)$$

We can say that it discontinuous at $x = 2$