

Discrete Mathematics

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IT241

Lecture 7



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Propositional & predicate logic

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LEARNING OUTCOMES

- ◉ Understand the basic concepts used in propositional logic
- ◉ Construct truth tables to evaluate the truth value of compound propositions.
- ◉ Understand the concepts of predicate logic, including variables, predicates, quantifiers
- ◉ Applications of propositional and predicate logic



LECTURE OUTLINE

- ◉ Propositional logic
 - ◉ Propositional variables
 - ◉ Logical connectives (AND, OR, NOT)
 - ◉ Truth values
 - ◉ Logical Equivalences
- ◉ Predicate Logic
 - ◉ The Language of Quantifiers
 - ◉ Logical Equivalences
- ◉ Valid Argument

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
 - a) The Moon is made of green cheese.
 - b) IT241 is an easy class.
 - c) $1 + 0 = 1$
 - d) $0 + 0 = 2$

- Examples that are **not propositions**.
 - a) Sit down!
 - b) What time is it?
 - c) $x + 1 = 2$
 - d) $x + y = z$

- Constructing Propositions
 - Propositional Variables: $p, q, r, s, \dots$
 - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.

- **Compound Propositions**; constructed from logical connectives and other propositions
 - **Negation** $\neg$
 - **Conjunction** $\wedge$
 - **Disjunction** $\vee$
 - **Implication** $\rightarrow$
 - **Biconditional** $\leftrightarrow$

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- Example:** If p denotes “The earth is round.”, then $\neg p$ denotes “It is not the case that the earth is round,” or more simply “The earth is not round.”

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \wedge q$ denotes “I am at home and it is raining.”

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \vee q$ denotes “I am at home or it is raining.”

The Connective Or in Natural Languages(1)

- In English “or” has two distinct meanings.
 - “Inclusive Or” :
 - In the sentence “Students who have taken IT241 or IT120 may take this class” we assume that students need to have taken **one** of the prerequisites, but may have taken **both**. This is the meaning of disjunction.
 - For $p \vee q$ to be **true**, either one or both of p and q must be true.

The Connective Or in Natural Languages(2)

- “Exclusive Or” (Xor): “Either it will rain or I will play football”
- In $p \oplus q$, **one** of p and q must be true, **but not both**.
- The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as “if p , then q ”.
- **Example:** If p denotes “I am at home.” and q denotes “It is raining.” then $p \rightarrow q$ denotes “If I am at home then it is raining.”
- In $p \rightarrow q$, p is the *hypothesis (antecedent or premise)* and q is the *conclusion (or consequence)*.

Logical Connectives : Implication

- p ="I am hungry"
- q ="I will eat"
- **If** I am hungry **then** I will eat.

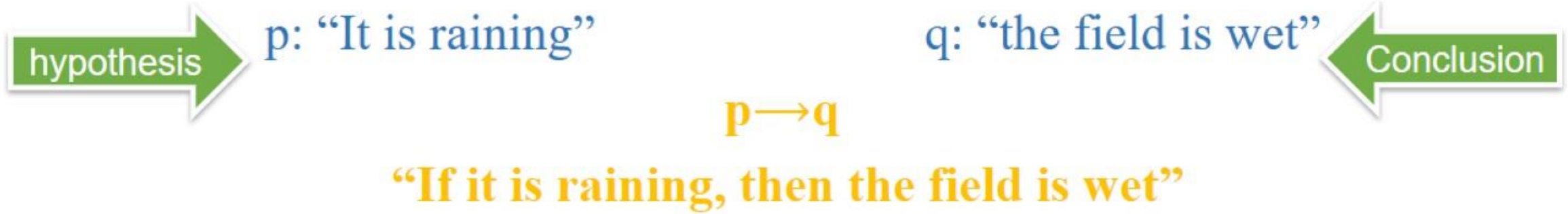
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
 - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$
 - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
 - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$

Converse, Contrapositive, and Inverse

- **Example:** Find the converse, inverse, and contrapositive of



Converse, Contrapositive, and Inverse

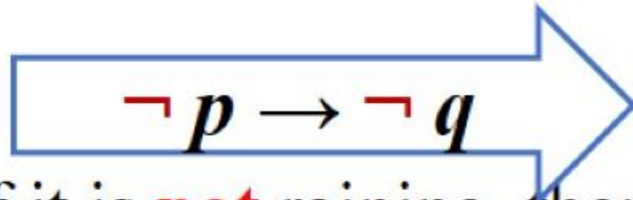
- **Example:** Find the converse, **inverse**, and contrapositive of

p : “It is raining”

q : “the field is wet”

$p \rightarrow q$ “If it is raining, then the field is wet”

Solution:


$$\neg p \rightarrow \neg q$$

- **Inverse:** If it is **not** raining, then the field is **not** wet.

Converse, Contrapositive, and Inverse

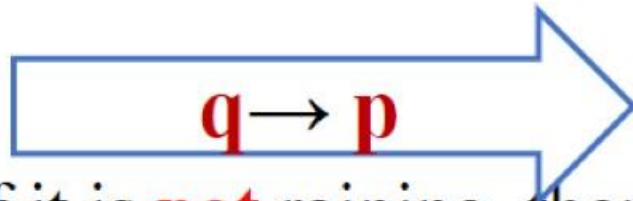
- **Example:** Find the **converse**, **inverse**, and contrapositive of

p : “It is raining”

q : “the field is wet”

$p \rightarrow q$ “If it is raining, then the field is wet”

Solution:



- **Inverse:** If it is **not** raining, then the field is **not** wet.
- **Converse:** If the field is wet, then it is raining.

Converse, Contrapositive, and Inverse

- **Example:** Find the **converse**, **inverse**, and **contrapositive** of
 p : “It is raining” q : “the field is wet”
 $p \rightarrow q$ “If it is raining, then the field is wet”

Solution:


$$\neg q \rightarrow \neg p$$

- **Inverse:** If it is **not** raining, then the field is **not** wet.
- **Converse:** If the field is wet, then it is raining.
- **Contrapositive:** if the field is **not** wet, then it is **not** raining

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, p iff q , read as “ p if and only if q .”

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes “I am at home.” and q denotes “It is raining.” then $p \leftrightarrow q$ denotes “I am at home if and only if it is raining.”

- Construction of a truth table:
- Rows
 - Need a row for every possible combination of values for the atomic propositions $= 2^n$.
- Columns
 - Need a column for the compound proposition (usually at far right)
 - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

- Two propositions are *equivalent* if they always have the *same truth* value.
- Example:** Show using a truth table that the conditional is equivalent to the contrapositive($\equiv$).

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

- Two propositions are *equivalent* if they always have the *same truth* value.
- Example:** Show using a truth table that the conditional is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

- How many rows are there in a truth table with n propositional variables?

Solution: 2^n

Precedence of Logical Operators

- $p \vee q \rightarrow \neg r$
- $(p \vee q) \rightarrow \neg r$

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$

If the intended meaning is $p \vee (q \rightarrow \neg r)$

then parentheses must be used.

Precedence of Logical Operators



- Translating English to Propositional Logic
- System Specifications
- Logic Circuits

- convert an English sentence to a statement in propositional logic
- “If I go to Harry’s or to the country, I will not go shopping.”
 - p : I go to Harry’s
 - q : I go to the country.
 - r : I will go shopping.

If p or q then not r .

$$(p \vee q) \rightarrow \neg r$$

Tautologies, Contradictions, and Contingencies

- A tautology is a proposition which is always true.
 - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
 - Example: $p \wedge \neg p$

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- This truth table shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$.
- Two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a **tautology**.
- Two compound propositions p and q are equivalent **if and only if** the columns in a truth table giving their truth values **agree**.
- We write this as $p \leftrightarrow q$ or as $p \equiv q$ where p and q are compound propositions.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

- A tautology is a proposition which is always true.
 - Example: $p \vee \neg p$
- A *contradiction* is a proposition which is always false.
 - Example: $p \wedge \neg p$

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- Predicate logic(First-Order Logic (FOL)) uses the following new features:
 - Variables: x, y, z
 - Predicates: $P(x), M(x)$
 - Quantifiers:
- *Propositional functions* are a generalization of propositions.
 - They contain variables and a predicate, e.g., $P(x)$
 - Variables can be replaced by elements from their *domain*.

- The statement $P(x)$ is said to be the value of the propositional function P at x .
- $P(x)$ denote “ $x > 0$ ” and the domain be the integers. Then:
 $P(-3)$ is false.
 $P(0)$ is false.
 $P(3)$ is true.
- Often the domain is denoted by U . So in this example U is the integers.

Examples of Propositional Functions

- Let “ $x + y = z$ ” be denoted by $R(x, y, z)$ and U (for all three variables) be the integers. Find these truth values:

$R(2, -1, 5)$

$R(3, 4, 7)$

$R(x, 3, z)$

Solution:

False

True

Not a Proposition

Examples of Propositional Functions

- Now let “ $x - y = z$ ” be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

$Q(2, -1, 3)$

Solution: T

$Q(3, 4, 7)$

Solution: F

$Q(x, 3, z)$

Solution: Not a Proposition

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes “ $x > 0$,” find these truth values:
 - $P(3) \vee P(-1)$ **Solution:** T
 - $P(3) \wedge P(-1)$ **Solution:** F
 - $P(3) \rightarrow P(-1)$ **Solution:** F
 - $P(3) \rightarrow \neg P(-1)$ **Solution:** T
- Expressions with variables are not propositions and therefore do not have truth values. For example,
 - $P(3) \wedge P(y)$
 - $P(x) \rightarrow P(y)$

- The two most important quantifiers are:
 - *Universal Quantifier*, “For all,” symbol: $\forall$
 - *Existential Quantifier*, “There exists,” symbol: $\exists$
- We write as in $\forall x P(x)$ and $\exists x P(x)$.
- $\forall x P(x)$ asserts $P(x)$ is true for every x in the *domain*.
- $\exists x P(x)$ asserts $P(x)$ is true for some x in the *domain*.

- $\forall x P(x)$ is read as “For all x , $P(x)$ ” or “For every x , $P(x)$ ”

Examples:

- 1) If $P(x)$ denotes “ $x > 0$ ” and U is the integers, then $\forall x P(x)$ is **false**.
- 2) If $P(x)$ denotes “ $x > 0$ ” and U is the positive integers, then $\forall x P(x)$ is **true**.

- $\exists x P(x)$ is read as “For some x , $P(x)$ ”, or as “There is an x such that $P(x)$,” or “For at least one x , $P(x)$.”

Examples:

1. If $P(x)$ denotes “ $x > 0$ ”

1. U is the integers, then $\exists x P(x)$ is true.

2. It is also true if U is the positive integers.

2. If $P(x)$ denotes “ $x < 0$ ” and U is the positive integers, then $\exists x P(x)$ is false.

- Application in artificial intelligence: Propositional logic is used in knowledge representation and reasoning in artificial intelligence systems.
- System specifications
- Logic Gates

All cats have tails.

Fluffy is a cat.

Therefore, Fluffy has tail.

- We can express the premises (above the line) and the conclusion (below the line) in predicate logic as an argument:

$$\forall x(Man(x) \rightarrow Mortal(x))$$

$$Man(Socrates)$$

$$\therefore Mortal(Socrates)$$

- An *argument* in propositional logic is a sequence of propositions *premises* & *conclusion*.
- The argument is valid if the premises imply the conclusion

Recap

Which of these sentences are propositions?

- a) Cairo is the capital of Egypt.
- b) Miami is the capital of Florida.
- c) $2 + 3 = 5$.
- d) $5 + 7 = 10$.
- e) $x + 2 = 11$.
- f) Answer this question.

Solution: **a) Yes b) Yes c) Yes d) Yes e) No f) No**

Express each of These English sentences as compound propositions

p: “Swimming at the New Jersey shore is allowed”

q : “Sharks have been spotted near the shore

- a) Sharks have **not** been spotted near the shore.
- b) Swimming at the New Jersey shore is allowed, and sharks have been spotted near the shore.
- c) Swimming at the New Jersey shore is **not** allowed, or sharks have been spotted near the shore.
- d) If swimming at the New Jersey shore is allowed, then sharks have **not** been spotted near the shore

a) $\neg q$ b) $p \wedge q$ c) $\neg p \vee q$ d) $p \rightarrow \neg q$

p: “Swimming at the New Jersey shore is allowed **q** : “Sharks have been spotted near the shore

e) If sharks have **not** been spotted near the shore, then swimming at the New Jersey shore is allowed.

f) If swimming at the New Jersey shore is **not** allowed, then sharks have **not** been spotted near the shore.

g) Swimming at the New Jersey shore is allowed if and only if sharks have **not** been spotted near the shore.

h) Swimming at the New Jersey shore is **not** allowed, and either swimming at the New Jersey shore is allowed or sharks have **not** been spotted near the shore.

Solution e) $\neg q \rightarrow p$ f) $\neg p \rightarrow \neg q$ g) $p \leftrightarrow \neg q$ h) $\neg p \wedge (p \vee \neg q)$

Determine whether each of these conditional statements is true or false.

- a) If $1 + 1 = 2$, then $2 + 2 = 5$.
- b) If $1 + 1 = 3$, then $2 + 2 = 4$.
- c) If $1 + 1 = 3$, then $2 + 2 = 5$.
- d) If monkeys can fly, then $1 + 1 = 3$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Solution. a) False b) True c) True d) True

What is the **inverse** of the statement

“If two triangles are not similar, their corresponding angles are not congruent”?

- 1) If two triangles are similar, their corresponding angles are not congruent.
- 2) If corresponding angles of two triangles are not similar.
- 3) If two triangles are similar, their corresponding
- 4) If corresponding angles of two triangles are congruent, the triangles are similar.

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$
 $q \rightarrow p$ is the converse of $p \rightarrow q$
 $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

Solution : 3

What is the **inverse** of the statement

“If Mike did his homework, then he will pass this test”?

- 1) If Mike passes this test, then he did his homework.
- 2) If Mike does pass this test, then he did not do his homework.
- 3) If Mike did not do his homework, then he will not pass this test.
- 4) If Mike did not do his homework, then he will pass this test.

Solution : 3

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$
 $q \rightarrow p$ is the converse of $p \rightarrow q$
 $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

What is the **inverse** of the statement

“If Roba works hard, then she succeeds”?

- 1) If Roba succeeds, then she works hard.
- 2) If Roba does not succeed, then she does not work hard.
- 3) If Roba works hard, then she does not succeed.
- 4) If Roba does not work hard, then she does not succeed

Solution : 4

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$
 $q \rightarrow p$ is the converse of $p \rightarrow q$
 $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

What is the **inverse** of the statement

“If Roba works hard, then she succeeds”?

- 1) If Roba succeeds, then she works hard.
- 2) If Roba does not succeed, then she does not work hard.
- 3) If Roba works hard, then she does not succeed.
- 4) If Roba does not work hard, then she does not succeed.

Solution : 4

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$

$q \rightarrow p$ is the converse of $p \rightarrow q$

$\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

What is the **converse** of

“If an angle measures 90 degrees, then it is a right angle”?

- 1) If an angle is a right angle, then it measures 90 degrees
- 2) An angle is a right angle if it measures 90 degrees
- 3) If an angle is not a right angle, then it does not measure 90 degrees
- 4) If an angle does not measure 90 degrees, then it is not a right angle

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$
 $q \rightarrow p$ is the converse of $p \rightarrow q$
 $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

Solution : 1

What is the **contrapositive** of

“If I am tall, then I will bump my head”?

- 1) If I bump my head, then I am tall.
- 2) If I do not bump my head, then I am tall.
- 3) If I am tall, then I will not bump my head.
- 4) If I do not bump my head, then I am not tall.

Solution : 4

$\neg p \rightarrow \neg q$ is the inverse of $p \rightarrow q$
 $q \rightarrow p$ is the converse of $p \rightarrow q$
 $\neg q \rightarrow \neg p$ is the contrapositive of $p \rightarrow q$

True or false

$$\neg p \vee q \equiv \neg p \rightarrow q$$

p	q	$\neg p$	$\neg p \vee q$	$\neg p \rightarrow q$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

Solution False

1. Let $P(x)$ denote the statement “ $x \leq 4$.” What are these truth values?

a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$

2. Let $P(x)$ be the statement “The word x contains the letter a .” What are these truth values?

a) $P(\text{orange})$ **b)** $P(\text{lemon})$

Solution

1. T T F

2. T F



LECTURE SUMMARY

- ◉ Propositional logic: propositions, logical connectives, truth values
- ◉ Equivalence
- ◉ Tautologies , contradictions
- ◉ Predicate logic :Variables, Predicates, and quantifiers
- ◉ Valid argument

*Thank
you*

