

Digital Systems- ITC131

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Lecture 04: Boolean Algebra – part2

Contents

- Why study Boolean Algebra?
- Binary Logic and Gates
- Boolean Algebra
- Canonical and Standard Forms

Explanatory Example

- It is required to design a simple system to control a lamp at a room. The lamp should go on when there is someone in the room and we are at night. Else, if we are in the morning or no one in the room the lamp should go off.
- **Solution\Steps:**
 - Truth tables are used to represent the requirements for the system to be designed. It clarifies what is the desired output for the different states of the inputs.
 - From the truth table we can define the Boolean equation, simplify it (using Boolean algebra) then describe the logic diagram (implementation)

Explanatory Example

Solution:

We have two inputs and one output to control the lamp:

$X = 1$ when someone in the room

0 otherwise

$Y = 1$ in the morning

0 at night

Truth table		
X	Y	Z
0	0	0
0	1	0
1	0	1
1	1	0

Canonical Forms

- There are Two methods to form Boolean functions from truth table:

1) Sum of minterms

2) Product of maxterms

which called **canonical forms**.

Explanatory Example

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1	1	0

What is Minterm?

- **Minterms** are **AND** terms with every variable present in either true or complemented form (**standard product term**)
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., x'), there are 2^n minterms for n variables.
- **Example:** Two variables (X and Y) produce $2^2 = 4$ combinations:
 - (both normal)
 - (X normal, Y complemented)
 - (X complemented, Y normal)
 - (both complemented)
- Thus there are four minterms of two variables.

What is Maxterm?

- **Maxterms** are OR terms with every variable in true or complemented form. (standard sum term)
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., x'), there are 2^n maxterms for n variables.
- **Example:** Two variables (X and Y) produce $2^2 = 4$ combinations:
 - (both normal)
 - (x normal, y complemented)
 - (x complemented, y normal)
 - (both complemented)

3 variables Minterm?

Minterm (standard product term)

- Each combination in truth table corresponds to a minterm.
- 3 variables can form 2^3 minterm.
- An AND term that contains all literals.
 - **Primed** (complemented) if corresponding bit is 0.
 - **Unprimed** (not complemented) if corresponding bit is 1.
- Minterms are in the form m_i
- (what is i ?) $\rightarrow$ index

x	y	z	Minterms	
			Term	Designation
0	0	0	$x'y'z'$	m_0
0	0	1	$x'y'z$	m_1
0	1	0	$x'yz'$	m_2
0	1	1	$x'yz$	m_3
1	0	0	$xy'z'$	m_4
1	0	1	$xy'z$	m_5
1	1	0	xyz'	m_6
1	1	1	xyz	m_7

- All variables will be present in a minterm will be listed in the same order (usually alphabetically)

Boolean Algebra-part2

3 variables Maxterm

- **Maxterm (standard sum term)**
 - Each combination in truth table corresponds to a maxterm.
 - 3 variables can form 2^3 maxterm.
 - An **OR** term that contains all literals.
 - **Primed** (complemented) if corresponding bit is 1.
 - **Unprimed** (not complemented) if corresponding bit is 0.
 - Maxterms are in the form M_i
- All variables will be present in a maxterm will be listed in the same order (usually alphabetically)

x	y	z	Minterms		Maxterms	
			Term	Designation	Term	Designation
0	0	0	$x'y'z'$	m_0	$x + y + z$	M_0
0	0	1	$x'y'z$	m_1	$x + y + z'$	M_1
0	1	0	$x'yz'$	m_2	$x + y' + z$	M_2
0	1	1	$x'yz$	m_3	$x + y' + z'$	M_3
1	0	0	$xy'z'$	m_4	$x' + y + z$	M_4
1	0	1	$xy'z$	m_5	$x' + y + z'$	M_5
1	1	0	xyz'	m_6	$x' + y' + z$	M_6
1	1	1	xyz	m_7	$x' + y' + z'$	M_7

M and m Relationship

□ Each maxterm is the complement of its corresponding minterm, and **vice versa**

- $m_n' = M_n$ and Vice Versa

- For example:

- $000 = X' Y' Z' \rightarrow m_0 \rightarrow m_0' = (X' Y' Z')' = X + Y + Z = M_0$
- $001 = X' Y' Z \rightarrow m_1 \rightarrow m_1' = (X' Y' Z)' = X + Y + Z' = M_1$
- $010 = X' Y Z' \rightarrow m_2 \rightarrow m_2' = (X' Y Z')' = X + Y' + Z = M_2$
- $011 = X' Y Z \rightarrow m_3 \rightarrow m_3' = (X' Y Z)' = X + Y' + Z' = M_3$
- $100 = X Y' Z' \rightarrow m_4 \rightarrow m_4' = (X Y' Z')' = X' + Y + Z = M_4$
- $101 = X Y' Z \rightarrow m_5 \rightarrow m_5' = (X Y' Z)' = X' + Y + Z' = M_5$
- $110 = X Y Z' \rightarrow m_6 \rightarrow m_6' = (X Y Z')' = X' + Y' + Z = M_6$
- $111 = X Y Z \rightarrow m_7 \rightarrow m_7' = (X Y Z)' = X' + Y' + Z' = M_7$

Boolean Algebra-part2

Minterm/Maxterm- Four Variables

Index	Binary	Minterm	Maxterm
i	abcd	m_i	M_i
0	0000	$a' b' c' d'$	$a+b+c+d$
1	0001	$a' b' c' d$	$a+b+c+d'$
3	0011		
5	0101		
7	0111		
10	1010		
13	1101		
15	1111		

Canonical Forms

- A Boolean function expressed algebraically can be written in two ways **canonical** form.
- **Canonical forms** express all binary variables in every product (AND) or sum (OR) term of the Boolean function. There are two types of canonical form: **Sum of Minterms and Product of Maxterms**

Canonical Forms - SoM

Sum of Minterms (SoM)

- Given a truth table, a Boolean function can be expressed as sum of minterms by:
 - Forming a minterm for each combination of the variables that produces a 1 in the function and then taking the OR of all terms.

Canonical Forms - SoM

Example:

Given the truth table, express f_1 as SoM

$$f_1 = x'y'z + xy'z' + xyz \leftarrow \text{SoM Form}$$

$$= m_1 + m_4 + m_7$$

$$f_1(x,y,z) = \Sigma(m_1, m_4, m_7) \leftarrow \text{Shorthand SoM Form}$$

x	y	z	Function f_1
0	0	0	0
0	0	1	1 $x'y'z$
0	1	0	0
0	1	1	0
1	0	0	1 $xy'z'$
1	0	1	0
1	1	0	0
1	1	1	1 xyz

• Exercise1:

• Example: Find $F_1 = m_0 + m_1 + m_6 + m_7$

• $F = x' y' z' + x' y' z + x y z' + x y z$

X	Y	Z	F
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

• Exercise 2:

• $F(A, B, C, D) = m_2 + m_9 + m_{11} + m_{14}$

• $F(A, B, C, D) = A'B'CD' + AB'C'D + AB'CD + ABCD'$

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Canonical Forms - PoM

Product of Maxterms (PoM)

- Given a truth table, a Boolean function can be expressed as product of maxterms by:
- Forming a **maxterm** for each combination of the variables that produces a 0 in the function and then taking the AND of all those terms.
- Forming **minterms** of the complement of the function and then take the complement.

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Canonical Forms - PoM

Example

Given the truth table, express F1 as PoM

$$\begin{aligned}
 f1 &= (x+y+z)(x+y'+z)(x+y'+z') \\
 &\quad (x'+y+z')(x'+y'+z) \\
 &= M_0. M_2. M_3 .M_5. M_6 \leftarrow \text{PoM Form}
 \end{aligned}$$

$$f1(x,y,z) = \prod(M_0, M_2, M_3, M_5, M_6) \leftarrow$$

Shorthand SoM Form

x	y	z	Function f_1	
0	0	0	0	M_0
0	0	1	1	M_1
0	1	0	0	M_2
0	1	1	0	M_3
1	0	0	1	M_4
1	0	1	0	M_5
1	1	0	0	M_6
1	1	1	1	M_7

- Exercise1:
- Example: Find $F_1 = M_3 \cdot M_5 \cdot M_6$
- $F1 = (X+Y'+Z').(X'+Y+Z').(X'+Y'+Z)$

X	Y	Z	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

- Exercise 2:
- $F(A, B, C, D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B, C, D) = (A+B+C'+D').(A'+B+C+D).(A'+B+C'+D').(A'+B'+C'+D)$

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Canonical Forms-SoM

Example

Express $F = A+B'C$ as a sum of minterms.

1) Algebraically

$$\begin{aligned}
 F &= A+B'C \\
 &= A(B+B') + B'C = AB + AB' + B'C \\
 &= AB(C+C') + AB'(C+C') + (A+A')B'C \\
 &= ABC + ABC' + AB'C + AB'C' + A'B'C
 \end{aligned}$$

2) Using TT

- Build the truth table
 - Choose the minterms where F equals to 1.
 - Sum them together.

$$= m_1 + m_4 + m_5 + m_6 + m_7$$

$$F = \Sigma(1, 4, 5, 6, 7)$$

Truth Table for $F = A + B'C$

A	B	C	F	
0	0	0	0	
0	0	1	1	A'B'C
0	1	0	0	
0	1	1	0	
1	0	0	1	AB'C'
1	0	1	1	AB'C
1	1	0	1	AB'C
1	1	1	1	ABC

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Example

Express $F = A+B'C$ as a sum of minterms.

1) Algebraically

$$F = A+B'C$$

$$= A(B+B') + B'C = AB + AB' + B'C$$

$$= AB(C+C') + AB'(C+C') + (A+A')B'C$$

$$= ABC + ABC' + AB'C + AB'C' + A'B'C$$

2) Using TT

- Build the truth table
 - Choose the minterms where F equals to 1.
 - Sum them together.

$$= m_1 + m_4 + m_5 + m_6 + m_7$$

$$F = \Sigma(m_1, m_4, m_5, m_6, m_7)$$

Truth Table for $F = A + B'C$

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

A'B'C

AB'C'

AB'C

AB'C

ABC

Boolean Algebra-part2

Conversion Between Canonical Forms

- To convert from one canonical form to another, **interchange** the Symbols Σ and Π and **list missing numbers** from the original form.
- Example:** Consider the function $F = xy + x'z$
 - $F(x, y, z) = \Sigma(1, 3, 6, 7) \rightarrow \text{SoM}$
 - $F(x, y, z) = \Pi(0, 2, 4, 5) \rightarrow \text{PoM}$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

Function Complements ... cont..

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- **Example:** Given $F(X,Y,Z) = \sum 1,3,5,7$
- $F'(X,Y,Z) = \sum 0,2,4,6$
- $F'(X,Y,Z) = \prod 1,3,5,7$

Function Complements

- The complement of a function expressed as a product of Maxterms is constructed by selecting the Maxterms missing in the maxterm canonical forms.
- Alternatively, the complement of a function expressed by a product of Maxterms form is simply the sum of Minterms with the same indices.
- **Example:** Given $F(X,Y,Z) = \prod 1,3,5,7$
- $F'(X,Y,Z) = \prod 0,2,4,6$
- $F'(X,Y,Z) = \sum 1,3,5,7$

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Standard Forms

- In both canonical forms, each minterm or maxterm must contain, by definition, all the variables, either complemented or uncomplemented.
- Another way to express Boolean functions is in **standard** form. Where the terms that form the function may contain any number of literals.
- There are two types of standard forms:
 - **Sum of Products (SoP):**
 - a Boolean expression containing AND terms, called **product terms**.
 - The sum denotes the **Oring** of these terms. Example: $F_1 = y' + xy + x'yz'$
 - Represented by a group of AND gates followed by a single OR gate

Boolean Algebra-part2

Standard Forms

- Product of Sums (PoS):
 - A Boolean expression containing OR terms, called sum terms.
 - The product denotes the **Anding** of these terms.
 - Example: $F_2 = x(y' + z)(x' + y + z')$
 - Represented by a group of OR gates followed by a single AND gate

Standard Forms

- **Standard Sum-of-Products (SoP) form:** equations are written as an OR of AND terms
- **Standard Product-of-Sums (PoS) form:** equations are written as an AND of OR terms
- *Examples:*
 - **SOP:** $ABC + A'B'C + B$
 - **POS:** $(A+B).(A+B+C).C$
- These “mixed” forms are neither SOP nor POS
 - $(AB+C)(A+C)$
 - $ABC' + AC(A+B)$

Boolean Algebra-part2

Canonical(SoM) Vs. Standard (SoP)

What is the difference between SoM and SoP?

$$F(A,B,C) = \Sigma (1,4,5,6,7)$$

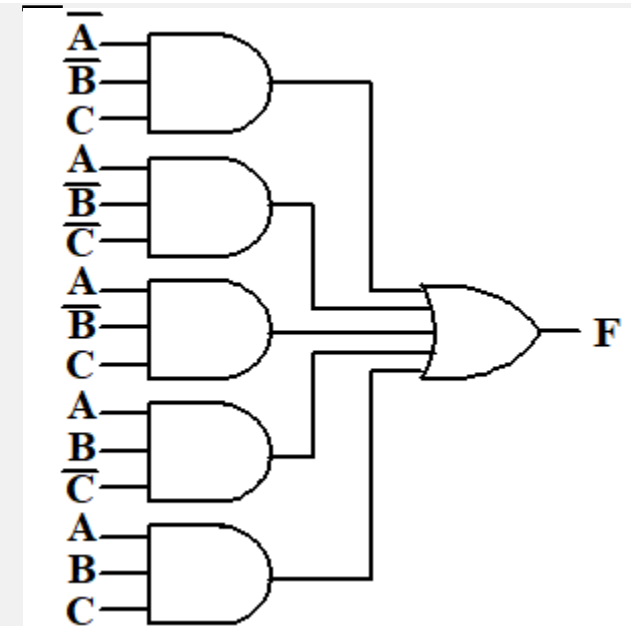
- Writing the minterm expression (SoM):

$$F = A'B'C + AB'C' + AB'C + ABC' + ABC$$

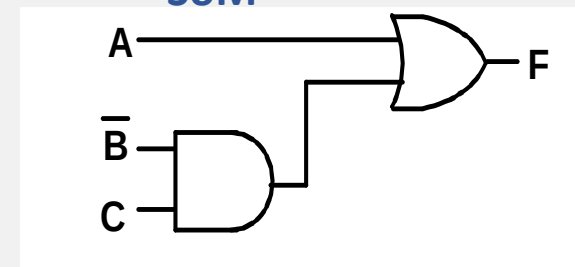
- Simplifying to (SoP):

$$F = B'C + A$$

The two implementations for F are shown here – it is quite apparent which is simpler!



SoM



SoP

SoM and SoP Observations

- The previous examples show that:
 - Simplified F (SoP) contains 3 literals compared to contains 15 in minterm F (SoM)
 - Canonical Forms (Sum-of-minterms), or other standard forms (SoP, PoS) differ in complexity
 - Boolean algebra can be used to manipulate equations into simpler forms.
 - Simpler equations lead to simpler two-level implementations

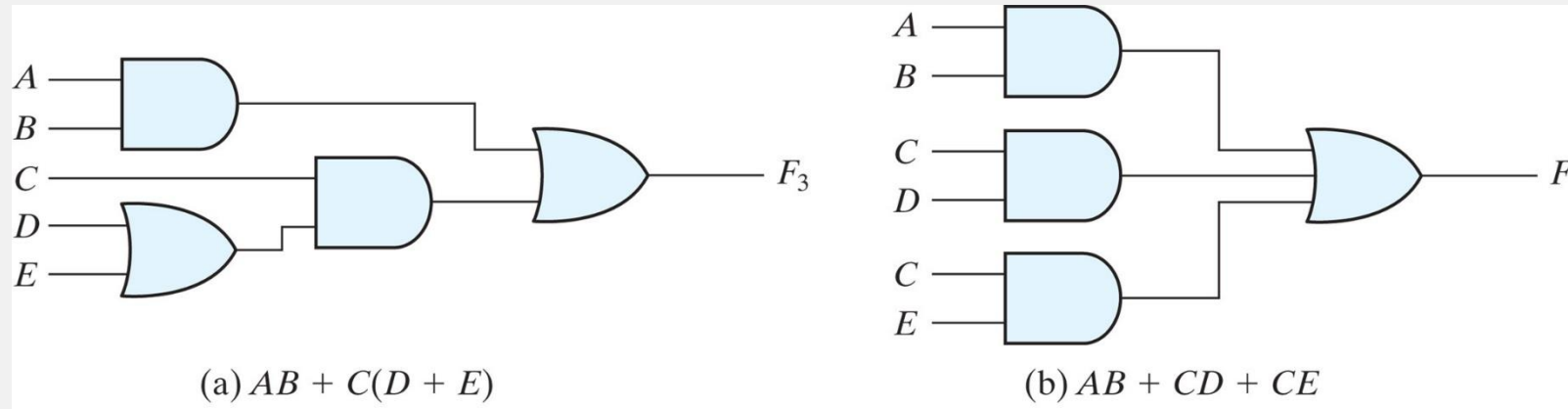
Non standard form

Example:

$$F=AB +C (D+E)$$

- It can be changed to a standard form by using distributive law.

$$F = AB + CD + CE$$



- Exercises

Convert each of the following to the other canonical form:

1. $F(x,y,z) = \Sigma(2,5,6)$

2. $F(A,B,C,D) = \Pi(0,1,2,4, 7, 9,12)$

- Solution:

1. $F(x,y,z) = \Pi(0,1,3,4,7)$

2. $F(A,B,C,D) = \Sigma(3,5, 6, 8, 10,11,13,14,15)$

Exercises

1. Express the following function as a sum of minterms and as a product of maxterms:

$$F(A, B, C) = B'.C + A'.C$$

- Solution:

- Using Boolean Algebra;

- $F = (A + A'). B'.C + (B + B'). A'.C$

- $F = A.B'.C + A'.B'.C + A'.B.C + A'.B'.C$

- $F = A'.B'.C + A'.B.C + A.B'.C = \sum m_1, m_3, m_5$ (Sum of Minterms)

- $F = \prod M_0, M_2, M_4, M_6, M_7$ (Product of Maxterms)

$$F(A, B, C) = B'.C + A'.C$$

- **Solution:**
- Using Truth Table (TT) :

$$F = \sum m_1, m_3, m_5$$

$$F = \prod M_0, M_2, M_4, M_6, M_7$$

A	B	C	A'	B'	B'.C	A'.C	F
0	0	0	1	1	0	0	0
0	0	1	1	1	1	1	1
0	1	0	1	0	0	0	0
0	1	1	1	0	0	1	1
1	0	0	0	1	0	0	0
1	0	1	0	1	1	0	1
1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0

Exercises

Express the following function as a Product of maxterms and as a sum of minterms:

$$F(X, Y, Z) = (X' + Y)(X + Z)$$

Solution:

Using Boolean algebra

$$F = (x' + y + z.z'). (x + z + y.y')$$

$$= (x' + y + z)(x' + y + z') (x + y + z)(x + y' + z)$$

$$= (x + y + z)(x + y' + z) (x' + y + z)(x' + y + z')$$

$$= M_0.M_2.M_4.M_5 = \prod M_0, M_2, M_4, M_5 \text{ (PRODUCT OF Maxterms)}$$

$$= \sum m_1, m_3, m_6, m_7 \text{ (Sum of Minterms)}$$

Exercises

Express the following function as a Product of maxterms and as a sum of minterms:

$$F(X, Y, Z) = (X' + Y)(X + Z)$$

Solution:

- Using Truth Tables (TT)

$$F = \prod M_0, M_2, M_4, M_5$$

$$F = \sum m_1, m_3, m_6, m_7$$

X	Y	Z	X'	(X'+Y)	(X+Z)	F
0	0	0	1	1	0	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	0	1	0
1	0	1	0	0	1	0
1	1	0	0	1	1	1
1	1	1	0	1	1	1

- Exercises:
- Given the function as defined in the truth table, express F using sum of minterms and product of maxterms, and express F' using sum of minterms and product of maxterms.

- Solution

- $F = m_1 + m_3 + m_4$

$$= \Sigma(1, 3, 4)$$

$$= x' y' z + x' y z + x y' z'$$

- $F = M_0 \cdot M_2 \cdot M_5 \cdot M_6 \cdot M_7$

$$= \Pi(0, 2, 5, 6, 7)$$

$$= (x + y + z) (x + y' + z) (x' + y + z') (x' + y' + z) (x' + y' + z')$$

x	y	z	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

• Exercises:

- Given the function as defined in the truth table, express F using sum of minterms and product of maxterms, and express F' using sum of minterms and product of maxterms.

• Solution

- F' as sum of 0-minterms:

$$F' = m_0 + m_2 + m_5 + m_6 + m_7$$

$$= \Sigma(0, 2, 5, 6, 7) = x' y' z' + x' y z' + x y' z + x y z' + x y z$$

x	y	z	F	F'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	1	0
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	0	1

- F' as product of 1-maxterms:

$$F' = M_1 \cdot M_3 \cdot M_4 = \Pi(1, 3, 4) = (x + y + z') (x + y' + z') (x' + y + z)$$

- Exercises

- Express the following Boolean function in a nonstandard form:

$$F(x,y,z) = x.y + x.z + y.z$$

Solution:

$$F = x y + (x + y) z$$

$$\text{or } = x (y + z) + y z =$$

$$\text{or } = x z + y (x + z)$$

End of Chapter 4