

Annuities & Sinking Funds

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Lecture 10



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MTH 112
Business Mathematics

In this lecture :

- * The compound interest problem is extended to annuities
- * Debt repayment
- * Sinking funds.
- * Excel for Financial Mathematics

Annuities

An annuity is a series of equal deposits made at equal intervals of time.

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i}$$

Annuities

And if the amount invested is sufficient to provide for the series of payments then

$$V_t = V_{ANU,t}$$
$$V_0(1+i)^t = A_0 \frac{(1+i)^t - 1}{i}$$

Solving for V_0 will give the required present value:

Annuities

And if the amount invested is sufficient to provide for the series of payments then

$$V_t = V_{ANU,t}$$

$$\frac{V_0(1+i)^t}{(1+i)^t} = A_0 \frac{(1+i)^t - 1}{i (1+i)^t}$$

Solving for V_0 will give the required present value:

$$\begin{aligned} V_0 &= \frac{A_0}{(1+i)^t} \cdot \frac{(1+i)^t - 1}{i} \\ &= A_0 \left[\frac{(1+i)^t}{(1+i)^t i} - \frac{1}{(1+i)^t i} \right] \\ &= A_0 \left[\frac{1}{i} - \frac{(1+i)^{-t}}{i} \right] \end{aligned}$$



Annuities

Hence

$$V_0 = A_0 \underbrace{\frac{1 - (1 + i)^{-t}}{i}}_{\text{annuity factor}} \quad (5.25)$$

where $\frac{1 - (1 + i)^{-t}}{i}$ is called the **annuity factor** or **cumulative present value factor**.

Example

To provide for future education costs, a family considers various methods of saving. Assume saving will continue for a period of 10 years at an interest rate of 7.5% per annum.

- (a) Calculate the value of the fund at the end of 10 years when a single deposit of £2000 is made annually.
- (b) How much should be deposited each year if the final value of the fund is £40000?
- (c) How much should be deposited each month if the final value of the fund is £40000?

(a) $A_0 = 2000$, $i = 0.075$, $t = 10$

$$\begin{aligned}
 V_{AMU,t} &= A_0 \frac{(1+i)^t - 1}{i} \\
 &= 2000 \frac{(1+0.075)^{10} - 1}{0.075} \\
 &= 2000 \frac{2.0610 - 1}{0.075} \\
 &= £28293.4
 \end{aligned}$$

The fund will be worth £28,293.4 at the end of 10 years.

(b)

$$V_{AND,10} = 40,000, \quad i = 0.075, \quad t = 10$$

$$40,000 = A_0 \frac{(1 + 0.075)^{10} - 1}{0.075}$$

$$\frac{40,000}{14.1467} = A_0 \quad \frac{14.1467}{14.1467}$$

$$A_0 = 2827.5145$$

To end up with £40,000, the annual deposit should be £2827.5145.

$$(C) \quad V_{ANO,10} = 40,000 \quad \frac{i}{12} = \frac{0.075}{12} = 0.00625$$

$$t \times 12 = 10 \times 12 = 120$$

$$40,000 = A_0 \frac{(1 + 0.00625)^{120} - 1}{0.00625}$$

$$\frac{40,000}{177.9303} = A_0 \frac{177.9303}{177.9303}$$

$$A_0 = 224.8071$$

Saving £ 224.8071 per month will also provide fund of £ 40,000 at the end of 10 years.



Present Value of Annuities

- (a) How much should you pay for an annuity of £1000 a year payable in arrears for 20 years, assuming an interest rate of 6%, if you are to break even?
- (b) How much should you pay for an annuity of £1000 a year payable in arrears quarterly for 20 years, assuming an interest rate of 6% per annum

Questions (a) and (b) are asking for the present value of the amount of the annuity: How much should be invested now in order to cover the required regular payments? The answers may be calculated by direct application of equation (5.25).

$$(a) \quad A_0 = 1000, \quad i = 0.06, \quad t = 20$$

$$\begin{aligned} V_0 &= A_0 \frac{1 - (1+i)^{-t}}{i} \\ &= 1000 \frac{(1 + 0.06)^{20} - 1}{0.06} \\ &= 11469.92 \end{aligned}$$

$$(b) \quad A_0 = \frac{1000}{4} = 250, \quad \frac{i}{4} = \frac{0.06}{4} = 0.015, \quad t \times 4 = 20 \times 4 = 80$$

$$V_0 = A_0 \frac{1 - (1 + \frac{i}{4})^{-t \times 4}}{\frac{i}{4}}$$

$$= (250) \left[1 - \frac{(1 + 0.015)^{-80}}{0.015} \right]$$

$$= (250) (46.4073)$$

$$= 11601.8309$$

Debt repayments

A loan is said to be **amortised** if both principal and interest are to be repaid by a series of equal payments made at equal intervals of time, assuming a fixed rate of interest throughout. For such a repayment scheme, the value of the loan (L) and interest rate are usually known but the amount to be repaid at each interval must be calculated.

$$L = \frac{A_0}{i} \left(1 - \frac{1}{(1+i)^t} \right)$$

Debt repayments

This equation is identical to (5.25).

(ii) Equation (5.26) may be solved for the size of the repayments, A_0 , given L , i and t :

$$A_0 = \frac{Li}{1 - \frac{1}{(1+i)^t}} = L \underbrace{\frac{i}{1 - (1+i)^{-t}}}_{\text{capital recovery factor}} \quad (5.28)$$

where $\frac{i}{1 - (1+i)^{-t}}$ is called the capital recovery factor.

Compare the capital recovery factor to the annuity factor (5.25) and you will observe that

$$\text{capital recovery factor} = \frac{1}{\text{annuity factor}}$$

Example

A mortgage of £200 000 is to be repaid over a 25-year period at a fixed interest rate of 4.5%.
Calculate the monthly repayments.

$$L = 200,000$$

$$\frac{i}{12} = \frac{0.0045}{12} = 0.00375$$

$$25 \times 12 = 300$$

$$A_0 = L \frac{\frac{i}{12}}{1 - (1 + \frac{i}{12})^{-t \times 12}}$$

$$= (200,000) \left[\frac{0.00375}{1 - (1.00375)^{-300}} \right]$$

$$= (200,000) (0.0055583)$$

$$= 1111.6650$$

Repayments per month.

How much of the repayment is interest?

How much of the repayment is interest?

When a loan is repaid by a series of equal instalments over equal intervals of time, the proportion of interest being repaid decreases steadily.

Example

A loan of £5000 is to be paid off in four equal payments at the end of each quarter.

Assuming an interest rate of 20%.

Calculate

- (a) the amount of each payment.
- (b) The present value of the loan.
- (c) The total amount of interest paid.
- (d) The amount of interest paid each quarter.

$$(a) \quad L = 5000 \quad i_{\frac{1}{4}} = \frac{0.20}{4} = 0.05, \quad 1 \times 4 = 4$$

$$A_0 = L \frac{i_{\frac{1}{4}}}{1 - (1 + i_{\frac{1}{4}})^{-t \times 4}}$$

$$= 5000 \left[\frac{0.05}{1 - (1.05)^{-4}} \right]$$

$$= \text{£}1410.06$$

(b) The present value of the loan is $V_0 = 5000$

$$V_0 = A_0 \frac{1 - (1 + \frac{i}{n})^{-t \times n}}{\frac{i}{n}}$$

$$= (1410.06) \left[\frac{1 - (1.05)^{-4}}{0.05} \right] = 5000$$

$$\begin{aligned} \text{c) Total payment} &= (1410.06)(4) \\ &= 5640.24 \end{aligned}$$

$$\begin{aligned} \text{Total interest} &= \text{total payments} - \text{value of loan} \\ &= 5640.24 - 5000 \\ &= \text{£}640.24 \end{aligned}$$

(d) At the start of quarter 1, the borrower has the loan of 5000 in his account.

$$\begin{aligned} \text{Value of start of quarter} + \text{interest of quarter} - \text{quarterly payments} \\ = \text{Value at end of quarter} \end{aligned}$$

Value at start (VSQ)	+ Interest (VSQ $\times$ 0.05)	– Payment (–1410.06)	Value at end (VSQ + VSQ $\times$ 0.05 – 1410.06)
5000.000	250.000	–1410.06	3839.940
3839.940	191.997	–1410.06	2621.877
2621.877	131.103	–1410.06	1342.920
1342.920	67.146	–1410.06	0.006
	640.246	–5640.24	

Sinking funds

A sinking fund is created by putting aside a fixed sum each year for the purpose of paying debts, replacing equipment, an annuity is set up repay the debt.

Fixed sum, A_0 , is set aside at the start of each year and interest is compounded annually at $i\%$, the fund will grow year as follows.

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Sinking funds

At the end of	Value of sinking fund
Year 1	$A_0(1+i) + A_0$
Year 2	$A_0(1+i)^2 + A_0(1+i) + A_0$
Year 3	$A_0(1+i)^3 + A_0(1+i)^2 + A_0(1+i) + A_0$
$\vdots$	$\vdots$
Year t	$A_0(1+i)^t + A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + \dots + A_0(1+i) + 0$
Value of sinking fund after t years	$A_0(1+i)^t + A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + \dots + A_0(1+i)$

$$P_0 = A_0$$

$$V_{SK,t} = A_0(1+i) \frac{(1+i)^t - 1}{(1+i) - 1} = A_0(1+i) \frac{(1+i)^t - 1}{i}$$

Example

A taxi service must replace cars every 5 years at a cost £450 000. At an 8% rate of interest
Calculate:

- (a) the size of the fund if £4000 is deposited at the beginning of each month.
- (b) the size of each quarterly payment necessary to meet this target.

$$\begin{aligned}
 (a) \quad A_0 &= 4000 & \frac{i}{12} &= \frac{0.08}{12} = 0.0067 & 5 \times 12 &= 60 \\
 V_{SK,t} &= A_0 (1 + \frac{i}{12}) \left[\frac{(1 + \frac{i}{12})^60 - 1}{\frac{i}{12}} \right] \\
 &= 4000 (1 + 0.0067) \left[\frac{(1.0067)^60 - 1}{0.0067} \right] \\
 &= 4000 (1.0067) (73.5522) \\
 &= 296,179.9989
 \end{aligned}$$

Not sufficient to meet target of £459,000.

(b) $V_{SK,t} = 450,000$, $i/4 = \frac{0.08}{4} = 0.02$, $5 \times 4 = 20$ Payment Interval

$$V_{SK,t} = A_0 (1 + i/4) \left[\frac{(1 + i/4)^{20} - 1}{i/4} \right]$$

$$450000 = A_0 (1.02) \left[\frac{(1.02)^{20} - 1}{0.02} \right]$$

$$\underline{450000} = A_0 \left(\frac{24.7833}{\underline{24.7833}} \right)$$



(b)

$$A_0 = \frac{450,000}{24.7833} = 18157.3882$$

Quarterly Payments of £18,157.3882 are required to meet the target £450,000.

Example

£1 is invested at a nominal rate of interest of 50%.

- (a) Set up a table comparing the growth of the investment over a 20-year period if interest is compounded
- (i) annually
 - (ii) quarterly
 - (iii) monthly
 - (iv) continuously
- (b) Graph the growth of the investment by all four methods given in (a) on the same diagram for the last five years.

Set up a table in Excel similar to Table 5.7. The formula for each method of compounding is entered into cells B4, C4, D4 and E4 respectively as follows:

(i) Annual compounding: $P_t = P_0(1 + i)^t$ for $t = 1$ to 20.

With the cursor (or pointer) in cell B4, type = B\$3*1.5^A4.

This formula states: principal $\times$ 1.5 raised to the power of time, when rate=0.5.

The \$ sign in front of the number 3 is required since we want to reference cell B3 (the same principal), when this formula is copied down column B. If the \$ sign is not used, then as the formula is copied down, B3 would become B4, B4 becomes B5, etc.

(Of course, with a £1 deposit, it is not necessary to reference any cell, but once the table is set up, the table will recalculate itself if the initial value of the investment is changed. Try it!)

Table 5.7 Different methods of compounding

	A	B	C	D	E
1			Method of compounding		
2	Year	Annually	Quarterly	Monthly	Continuous
3	0.00	1.00	1.00	1.00	1.00
4	1.00	1.50	1.60	1.65	1.65
5	2.00	2.25	2.57	2.66	2.72
6	3.00	3.38	4.11	4.35	4.48
7	4.00	5.06	6.58	7.10	7.39
8	5.00	7.59	10.55	11.58	12.18
9	6.00	11.39	16.89	18.90	20.09
10	7.00	17.09	27.06	30.85	33.12
11	8.00	25.63	43.34	50.35	54.60
12	9.00	38.44	69.42	82.17	90.02
13	10.00	57.67	111.20	134.11	148.41
14	11.00	86.50	178.12	218.88	244.69
15	12.00	129.75	285.31	357.23	403.43
16	13.00	194.62	457.02	583.03	665.14
17	14.00	291.93	732.05	951.56	1096.63
18	15.00	437.89	1172.60	1553.03	1808.04
19	16.00	656.84	1878.28	2534.70	2980.96
20	17.00	985.26	3008.65	4136.86	4914.77
21	18.00	1477.89	4819.27	6751.75	8103.08
22	19.00	2216.84	7719.55	11019.50	13359.73
23	20.00	3325.26	12365.22	17984.87	22026.47

$$P_t = P_0 (1+i)^t$$

$$= \boxed{B3} * 1.5 \wedge \boxed{A4}$$

$$= \boxed{C3} * 1.125 \wedge (\boxed{A4} * 4)$$

$$= \boxed{D3} * 1.041667 \wedge (\boxed{A4} * 12)$$

$$= \boxed{E3} * \exp(0.5 * \boxed{A4})$$

(ii) Quarterly compounding:

$$P_t = P_0 \left(1 + \frac{i}{4}\right)^{4t} \text{ for } t = 1 \text{ to } 20$$

With the cursor in cell C4, type = C\$3*1.125^(A4*4), then copy this formula down column C. This formula states: principal $\times$ (1 + rate/4) raised to the power of (time $\times$ 4), when rate = 0.5.

(iii) Monthly compounding:

$$P_t = P_0 \left(1 + \frac{i}{12}\right)^{12t} \text{ for } t = 1 \text{ to } 20$$

With the cursor in cell D4, type = D\$3*1.041667^(A4*12), then copy this formula down column D.

This formula states: principal $\times$ (1 + rate/12) raised to the power of (time $\times$ 12), when rate = 0.5

(iv) Continuous compounding: $P_t = P_0 e^{it}$

With the cursor in cell E4, type = E\$3*exp(0.5*A4), then copy this formula down column E. This formula states: principal $\times e$ raised to the power of (time $\times$ rate), when rate = 0.5

The results of the calculations are given in Table 5.7.

(b) Since we require the graph for the last 5 years only, copy the row of column titles to a new area on the sheet. Immediately below the row of titles copy the last

five rows of Table 5.7. The new table is given as Table 5.8. Now go through the usual steps of plotting the graph. The final graph should look like Figure 5.2.

From the graphs, the continuous method of compounding leads to the greatest cumulative total after 20 years, while the annual method gives the smallest sum. Furthermore, the longer the investment is left on deposit, the wider the differences in its value when compounded by methods (i), (ii), (iii) and (iv).

Table 5.8 Data for plotting Figure 5.2

Year	Annually	Quarterly	Monthly	Continuous
16.00	656.84	1 878.28	2 534.70	2 980.96
17.00	985.26	3 008.65	4 136.86	4 914.77
18.00	1 477.89	4 819.27	6 751.75	8 103.08
19.00	2 216.84	7 719.55	11 019.50	13 359.73
20.00	3 325.26	12 365.22	17 984.87	22 026.47

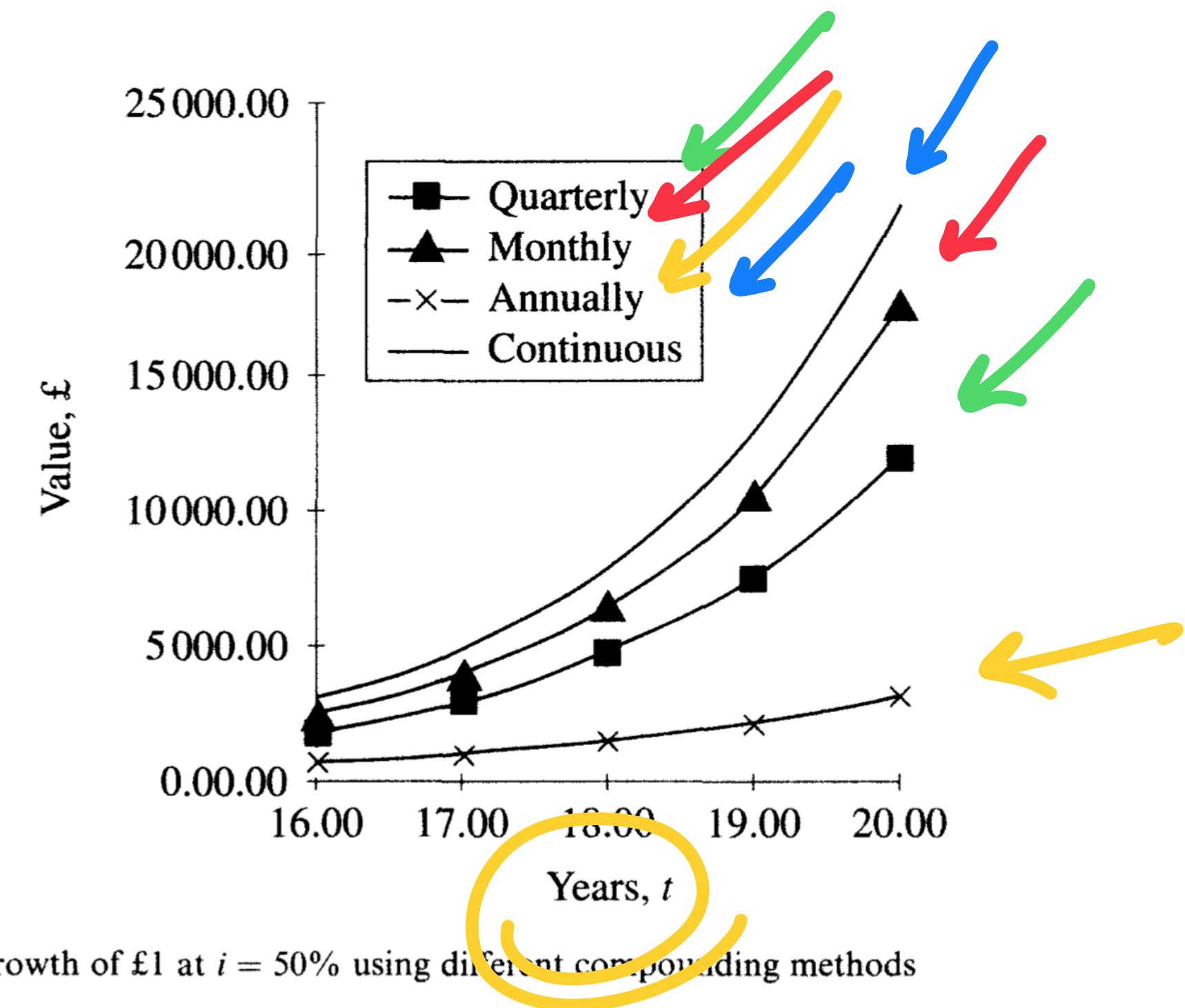


Figure 5.2 Growth of £1 at $i = 50\%$ using different compounding methods

Thank you