



مجموعة جامعة
الجميع الذكية
للتعليم القابضة

MIDOCEAN
UNIVERSITY
BY EVERYONE'S SMART UNIVERSITY

Mathematical Foundations For Data Science and AI

Lecture 02

AI102

Calculus; Properties of the derivatives

By
Dr. Khaled ELMenshawy

www.midocean.ae

Lecture

02

LEARNING OUTCOMES

- Understanding Trigonometric Functions and their derivatives.
- Understanding Inverse Function and its derivative.
- Understanding Euler's Number.
- Understanding Derivative of e^x .
- Understanding Logarithm and its derivative.
- Understanding Properties of the derivative; Sum rule, Product rule, and Chain rule.

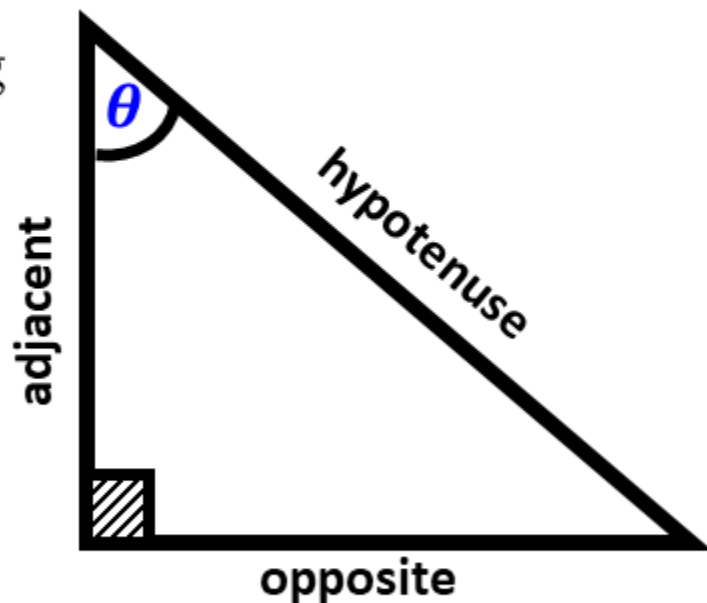
What is SohCahToa: Definition?

SohCahToa is a mnemonic device that is used to help remember how to calculate the angles and sides of the right triangle, using *trigonometric functions* sine, cosine, and tangent, where:

$$\text{SOH} = \sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \text{SOH} = \text{Sine is Opposite over Hypotenuse}$$

$$\text{CAH} = \cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \text{CAH} = \text{Cosine is Adjacent over Hypotenuse}$$

$$\text{TOA} = \tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} \quad \text{TOA} = \text{Tangent is Opposite over Adjacent}$$



Trigonometric functions sine, cosine, and tangent are important, they are used to calculate angles when sides are known and vice versa.

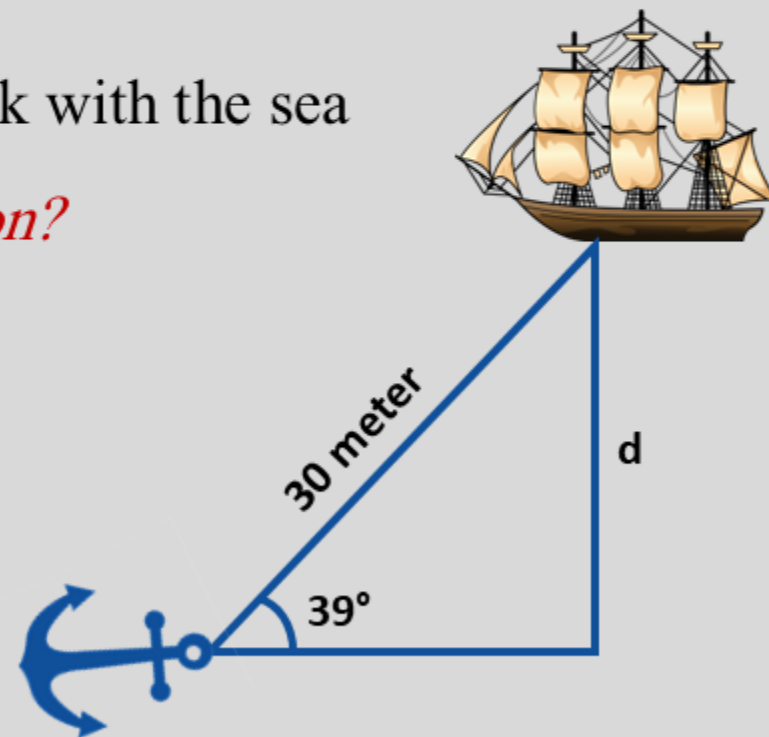
Real world example

The ship hook is 30 m down the sea, and the angle of the hook with the sea ground is 39° . *What is the deep of the sea at the ship's location?*

$$\sin 39 = \frac{d}{30}$$

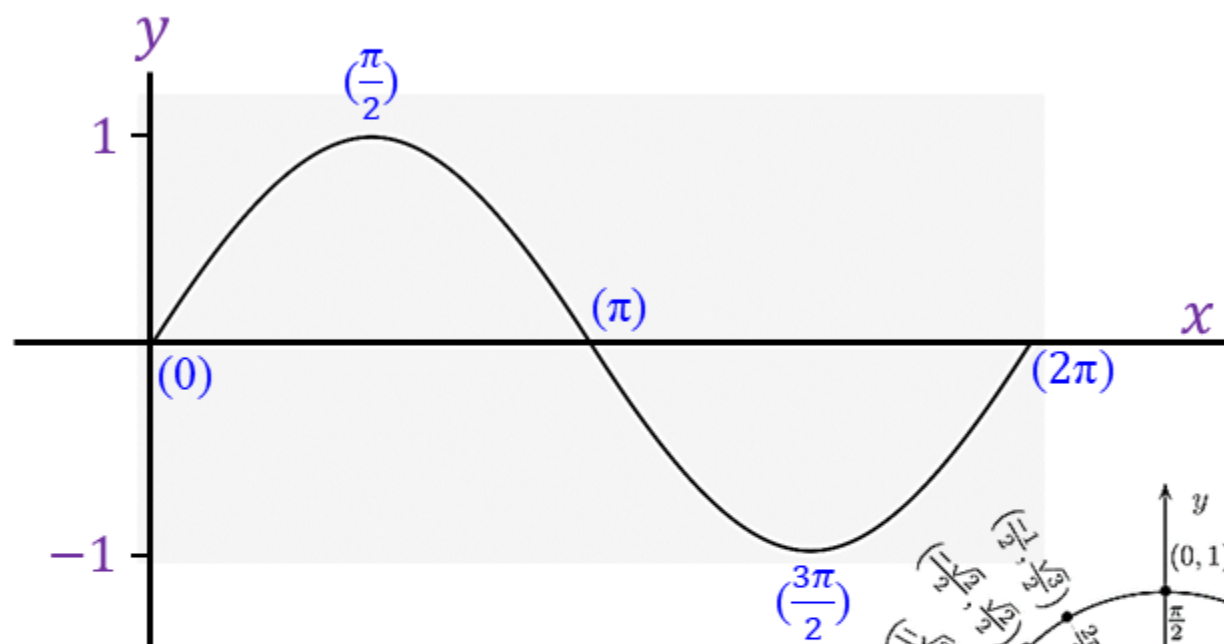
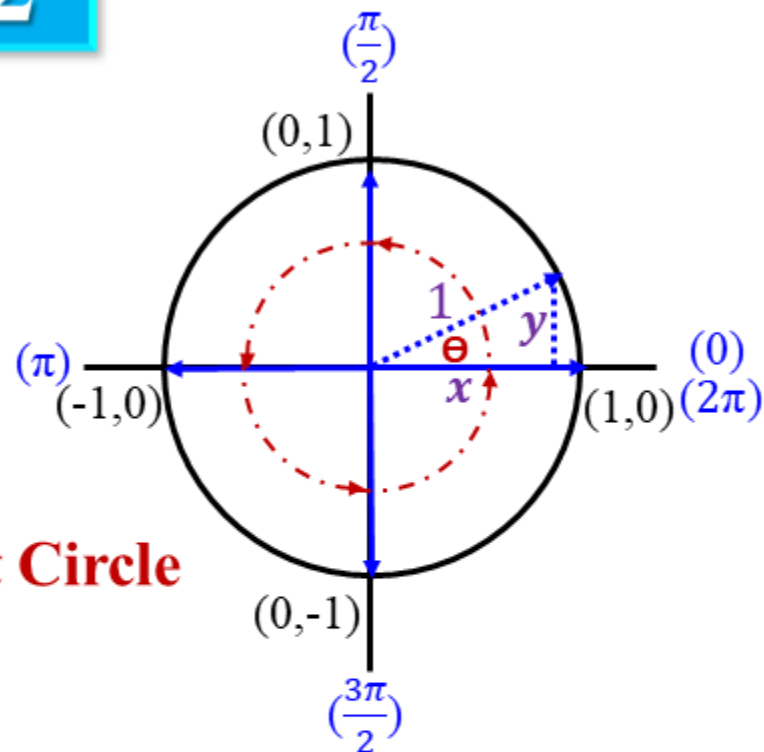
$$0.63 = \frac{d}{30}$$

$$d = 18.9 \text{ m}$$

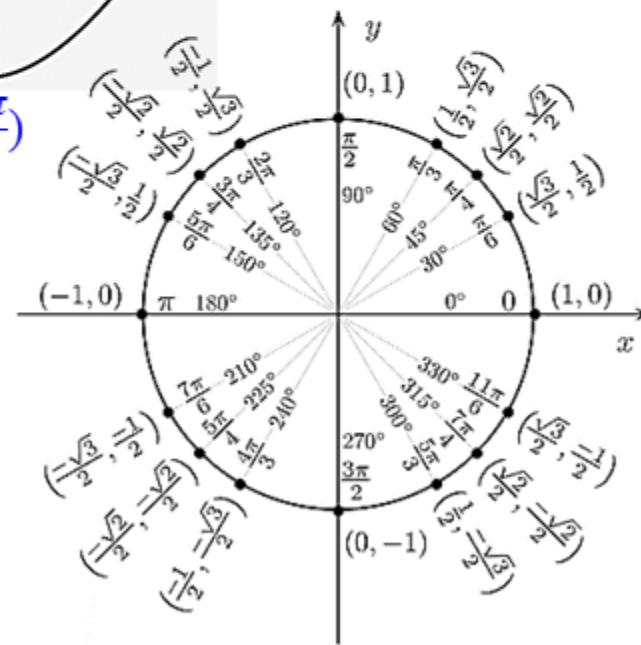


Trigonometric Functions

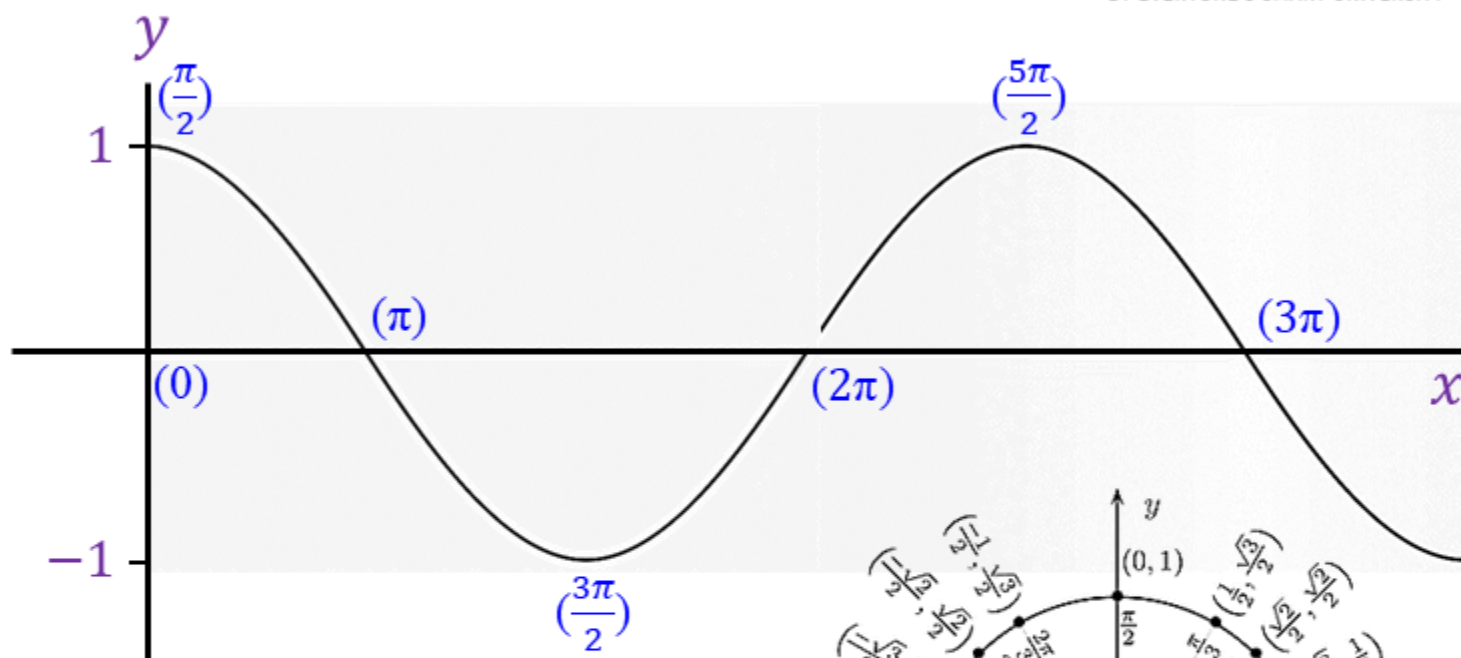
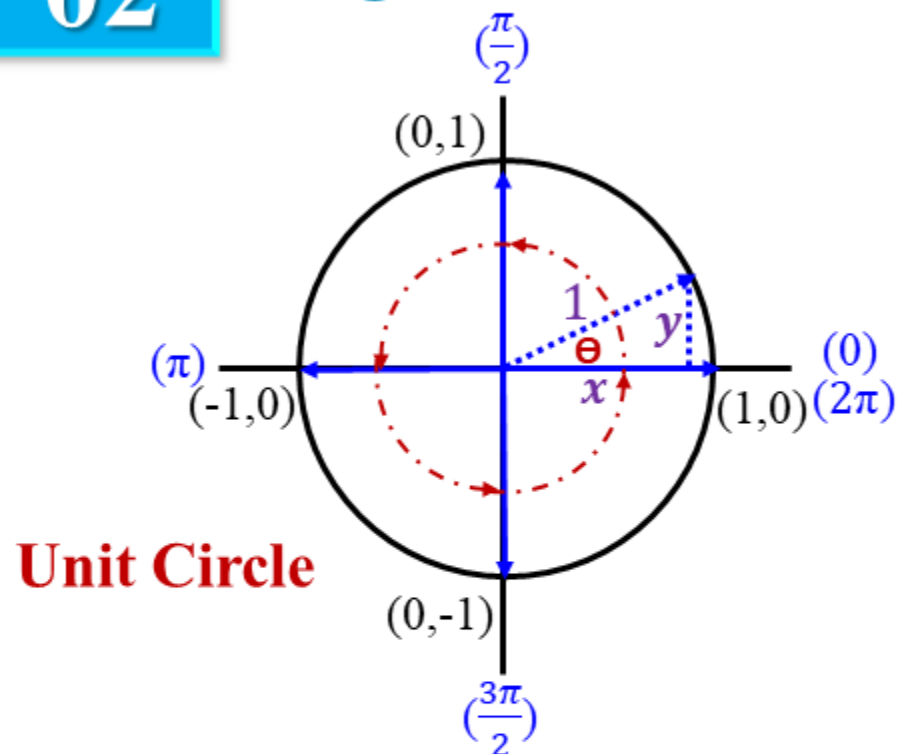
Unit Circle



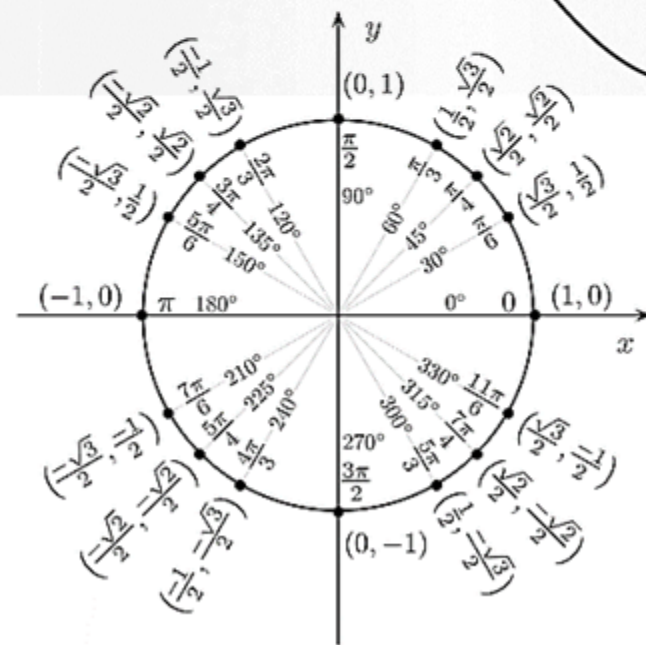
$$\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{1}$$



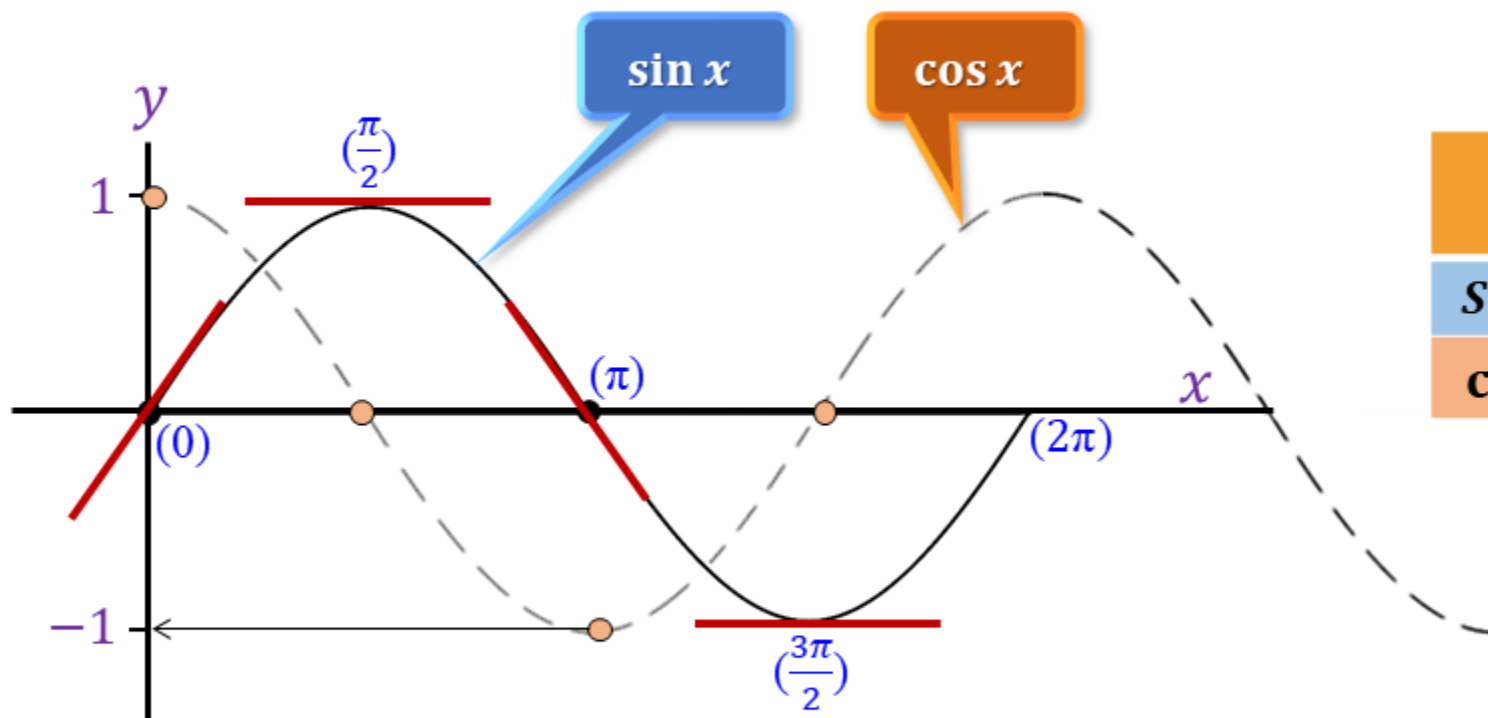
Trigonometric Functions



$$\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{1}$$



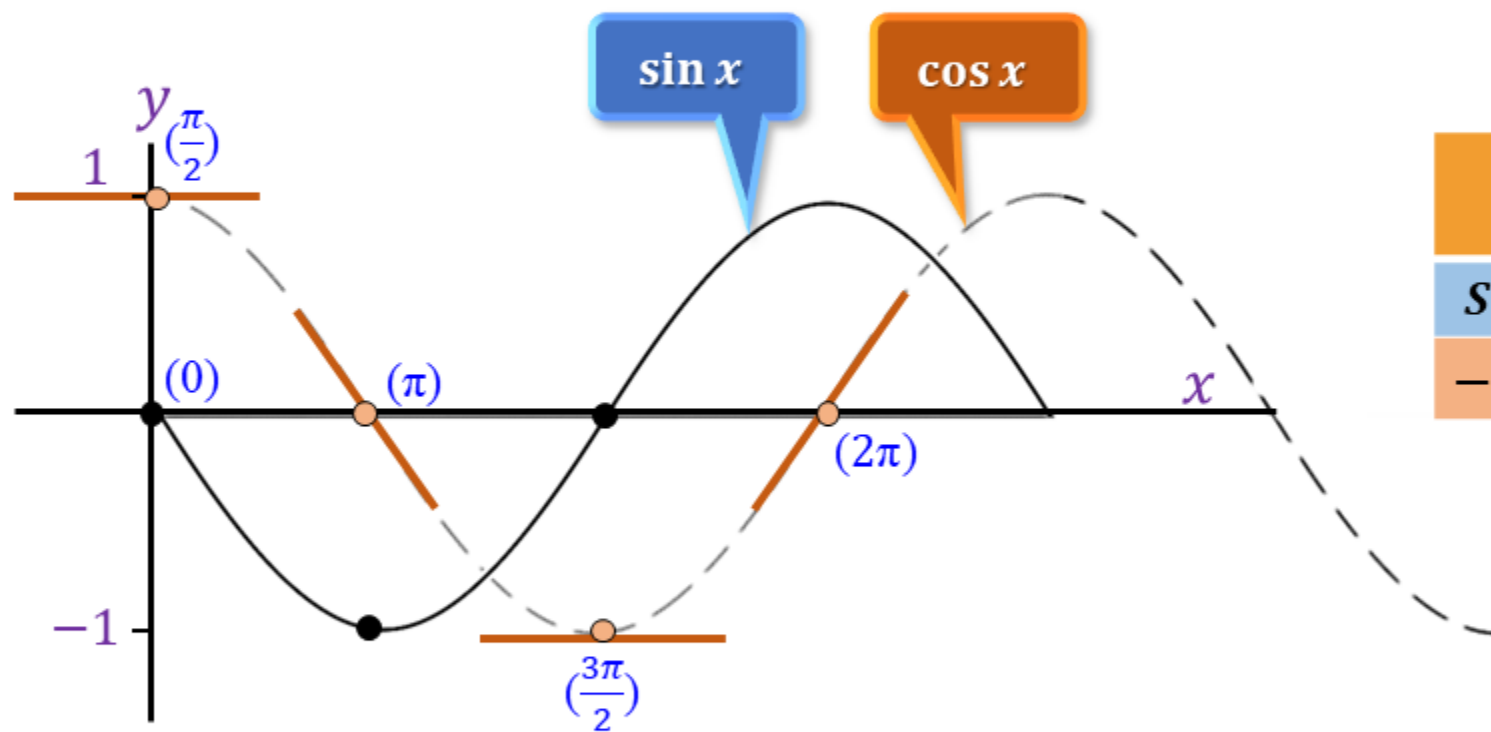
Derivative of Trigonometric Functions



x	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	0	π
<i>Slope</i>	0	0	1	-1
$\cos x$	0	0	1	-1

$$f(x) = \sin(x) \longrightarrow f'(x) = \cos(x)$$

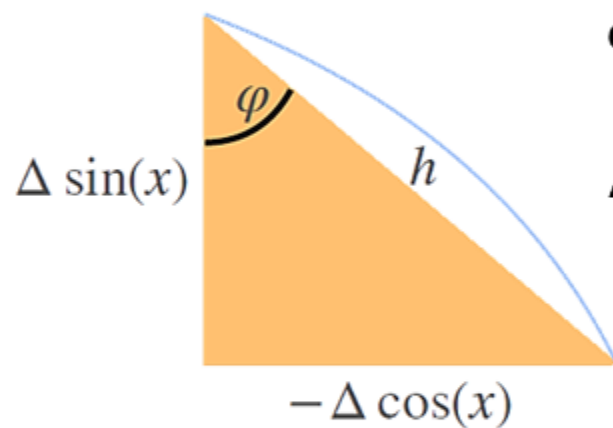
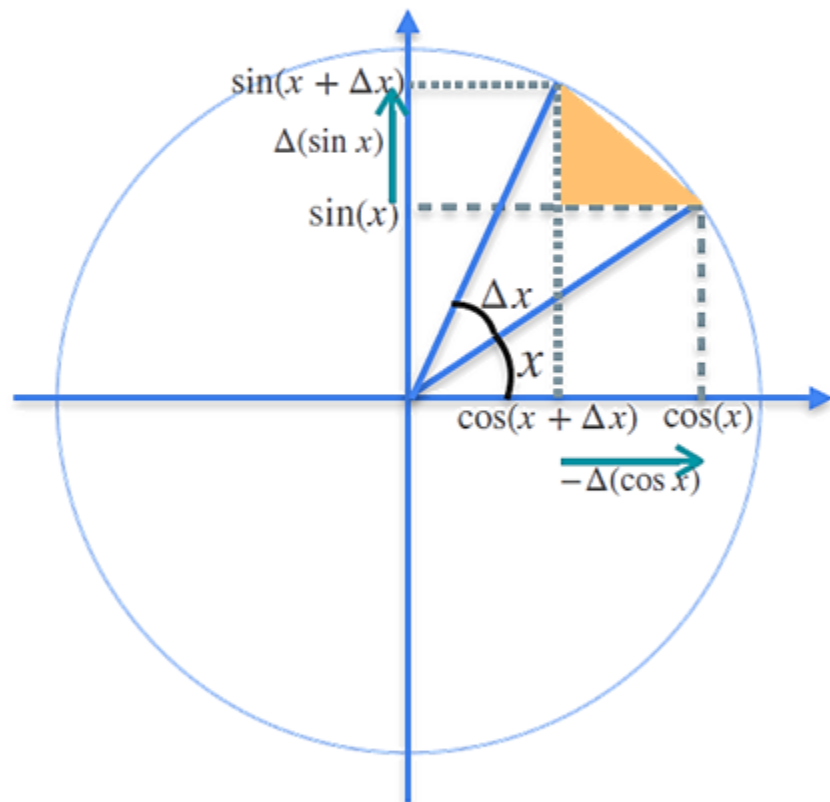
Derivative of Trigonometric Functions



x	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	0	π
<i>Slope</i>	0	0	1	-1
$-\sin x$	0	0	1	-1

$$f(x) = \cos(x) \longrightarrow f'(x) = -\sin(x)$$

Derivative of Trigonometric Functions



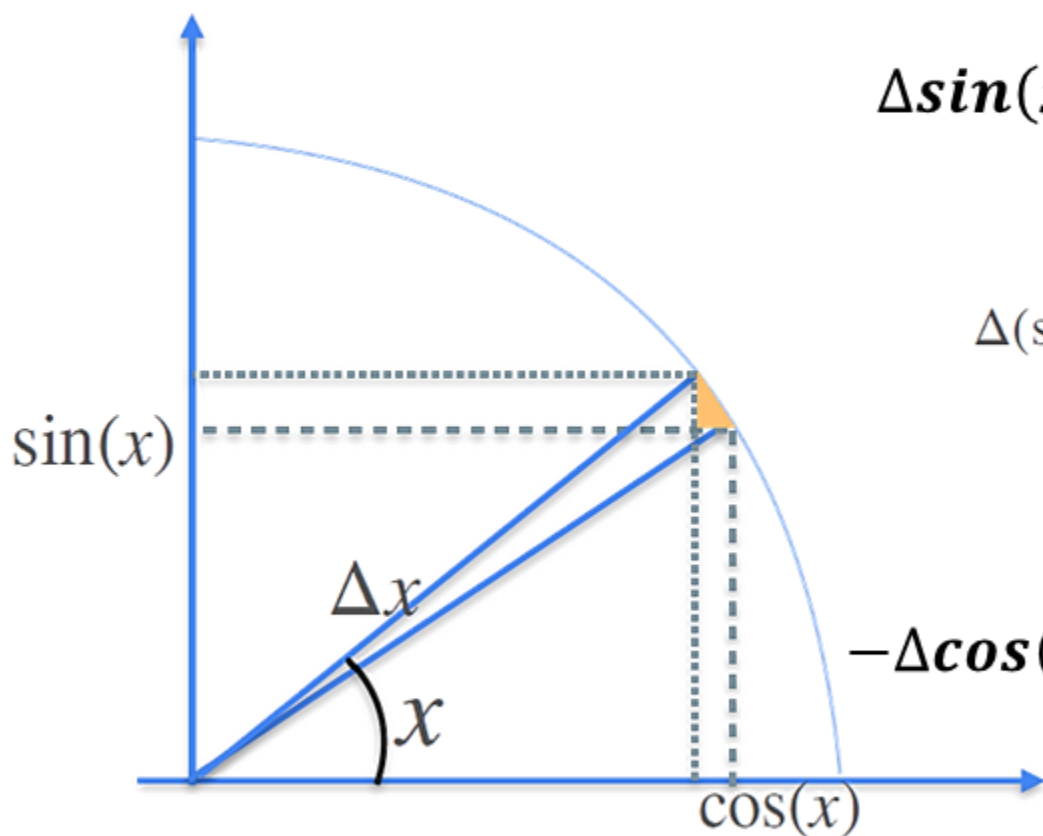
$$\cos \varphi = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{\Delta \sin(x)}{h}$$

$$\Delta \sin(x) = h \cos \varphi$$

$$\sin \varphi = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{-\Delta \cos(x)}{h}$$

$$-\Delta \cos(x) = h \sin \varphi$$

Derivative of Trigonometric Functions



$$\Delta \sin(x) = h \cos \phi \Rightarrow \Delta \sin(x) = \Delta x \cos(x)$$

$$f'(x) = \frac{d}{dx} \sin(x) = \cos(x)$$

$$-\Delta \cos(x) = h \sin \phi \Rightarrow -\Delta \cos(x) = \Delta x \sin(x)$$

$$f'(x) = \frac{d}{dx} \cos(x) = -\sin(x)$$

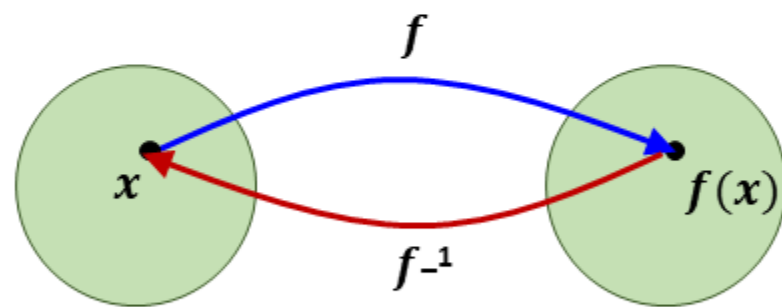
Inverse function is a function that is *the opposite of another function*. An inverse function does the exact opposite of the original function. The inverse of $f(x)$ is written mathematically as $f^{-1}(x)$, where the composition function $f(f^{-1}(x)) = x$

Consider the function $f(x) = \frac{x+3}{4}$

The inverse function $f^{-1}(x) = 4x - 3$

$$f(f^{-1}(x)) = 4x - 3 = 4\left(\frac{x+3}{4}\right) - 3 = x + 3 - 3 = x$$

$$\therefore f(f^{-1}(x)) = x$$



How to Find the Inverse of a Function?

1. The x and y values will switch in an inverse function, so replace the $f(x)$ notation with y .

$$f(x) = \sqrt{2x - 3} \Rightarrow y = \sqrt{2x - 3}$$

2. Next, replace every x in the equation with a y , and every y with an x .

$$y = \sqrt{2x - 3} \Rightarrow x = \sqrt{2y - 3}$$

3. Solve the new equation for y .

$$x^2 = (\sqrt{2y - 3})^2 \rightarrow x^2 = 2y - 3 \rightarrow 2y = x^2 + 3$$

4. Replace the y with the $f^{-1}(x)$ notation.

$$\therefore y = \frac{x^2 + 3}{2}$$

$$f^{-1}(x) = \frac{x^2 + 3}{2}$$

Examples

Find $f^{-1}(x)$ for the following functions:

$$f(x) = -3x - 7$$

$$y = -3x - 7$$

$$x = -3y - 7$$

$$-3y = x + 7$$

$$y = \frac{x + 7}{-3}$$

$$\therefore f^{-1}(x) = \frac{x + 7}{-3}$$

$$f(x) = \frac{1}{5}x + 8$$

$$y = \frac{1}{5}x + 8$$

$$x = \frac{1}{5}y + 8$$

$$y = 5(x - 8)$$

$$\therefore f^{-1}(x) = 5(x - 8)$$

Exercises

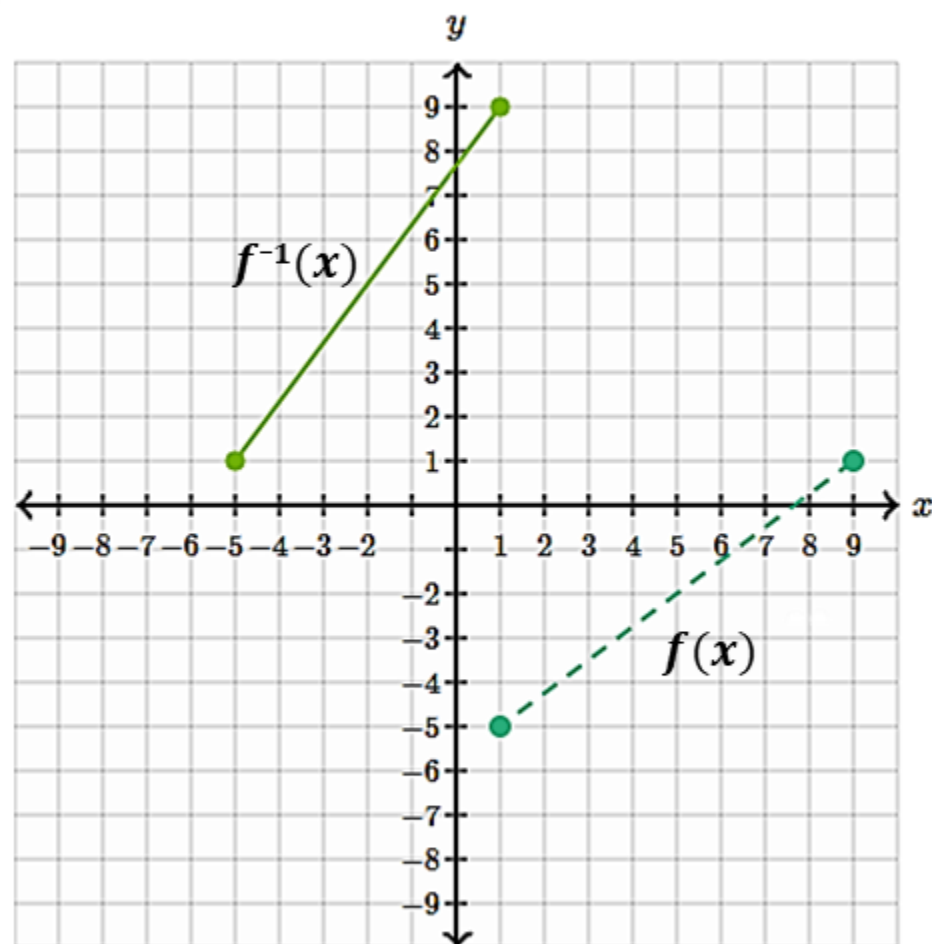
Find $f^{-1}(x)$ for the following functions:

$$f(x) = 2x + 1$$

$$f(x) = -\frac{2}{3}x - 5$$

$$f(x) = -\frac{1}{2}(x + 3)$$

Derivative of Inverse Function

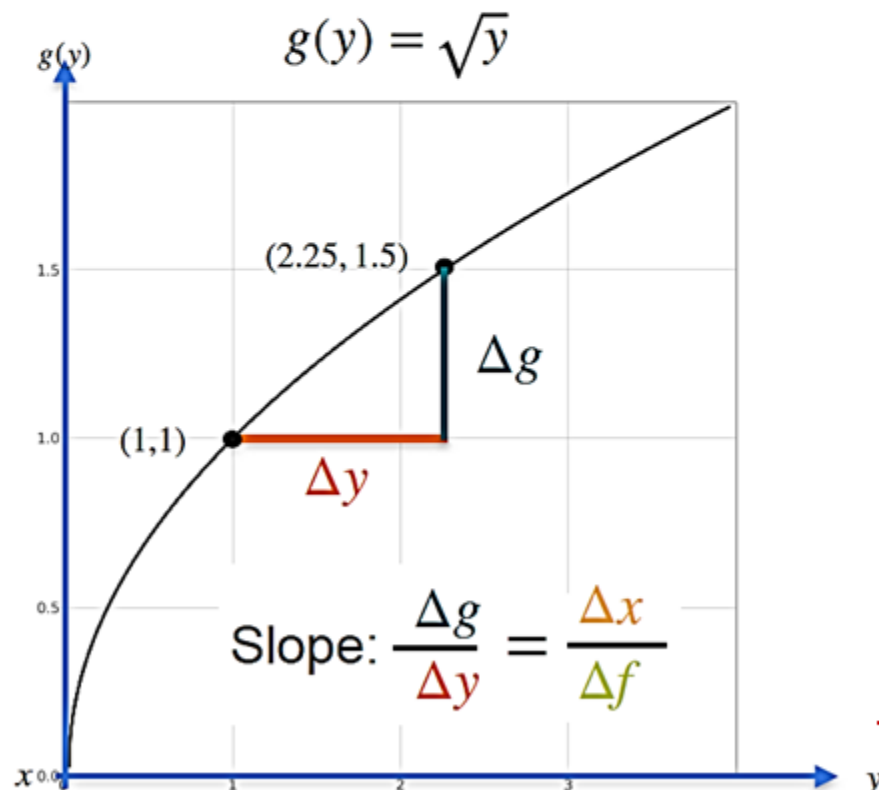
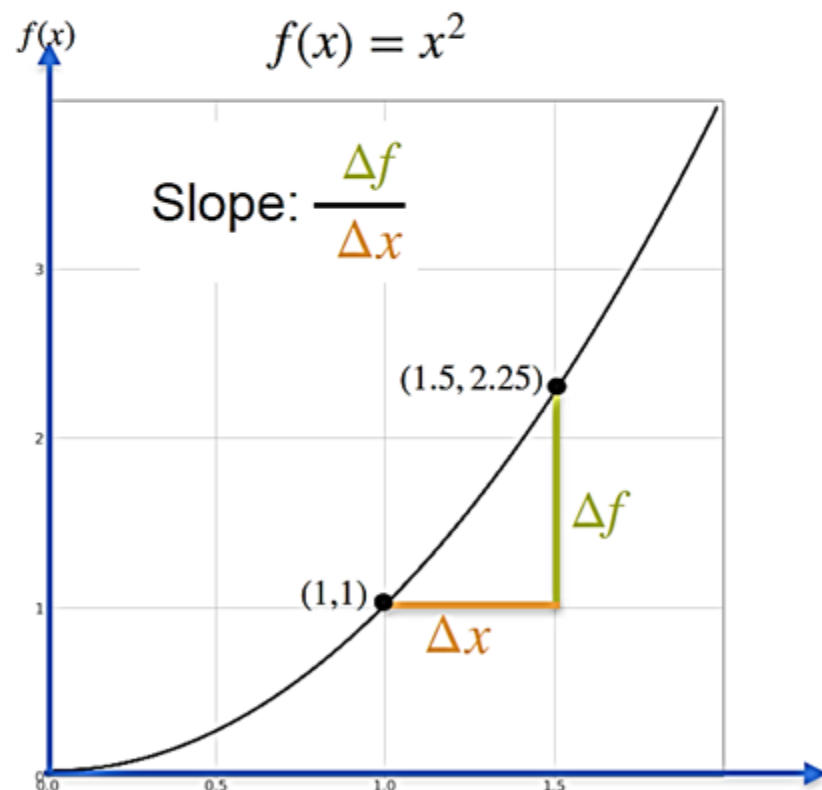


To find an inverse function the x 's and y 's *must be flipped*. This results in the inverse function graphs being the reflection of the original function over $x = y$. This also means the ordered pairs of the points in the inverse function are flipped.

Derivative of Inverse Function

Let $g(y)$ is the inverse function of $f(x)$, and $f(x) = x^2$, then $g(y) = \sqrt{y}$, according to the next graph:

$$f'(x) = \frac{\Delta f}{\Delta x}, \text{ and } g'(y) = \frac{\Delta g}{\Delta y}$$



$$\text{Since } \frac{\Delta f}{\Delta x} = \frac{\Delta y}{\Delta g}$$

$$\therefore \frac{\Delta f}{\Delta x} = 1 / \frac{\Delta g}{\Delta y}$$

$$\therefore f'(x) = 1 / g'(y)$$

$$\text{or } g'(y) = \frac{1}{f'(x)}$$



Leonard Euler

Euler's Number ' e ' is a numerical constant used in mathematical calculations.

The value of e is 2.718281828459045...so on. Just like π , e is also an *irrational number*. This is an important constant which is used in not only Mathematics but also in Physics.

There are a variety of ways to calculate Euler's number ' e '. The two most common ways are:

$$\blacksquare e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$$

Using a large value of n will produce an even more accurate approximation.

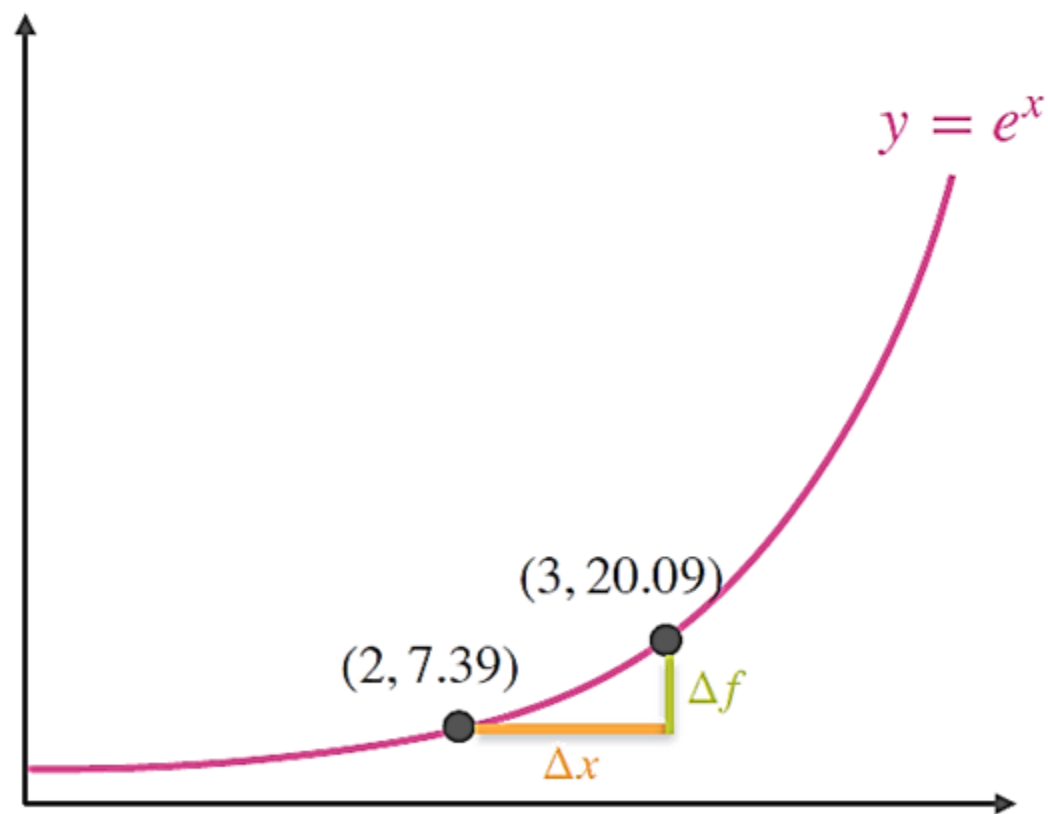
Using $n = 10,000$ yields

$$e = \left(1 + \frac{1}{10,000}\right)^{10,000} \cong 2.71815$$

The second method involves factorials, which are symbolized as an exclamation point. A factorial means multiply the number by every positive integer less than it. For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. It turns out that e is the infinite sum of $1/n!$ when n starts at 0 (keep in mind that $0! = 1$).

$$\blacksquare e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Using an upper limit of 10, i.e., summing the terms that result from plugging in $n = 0$ through $n = 10$, yields $e = \sum_{n=0}^{10} 1/n! \cong 2.71828$



In this case, $y = f(x) = e^x$, so $f'(x) = e^x$, thus:

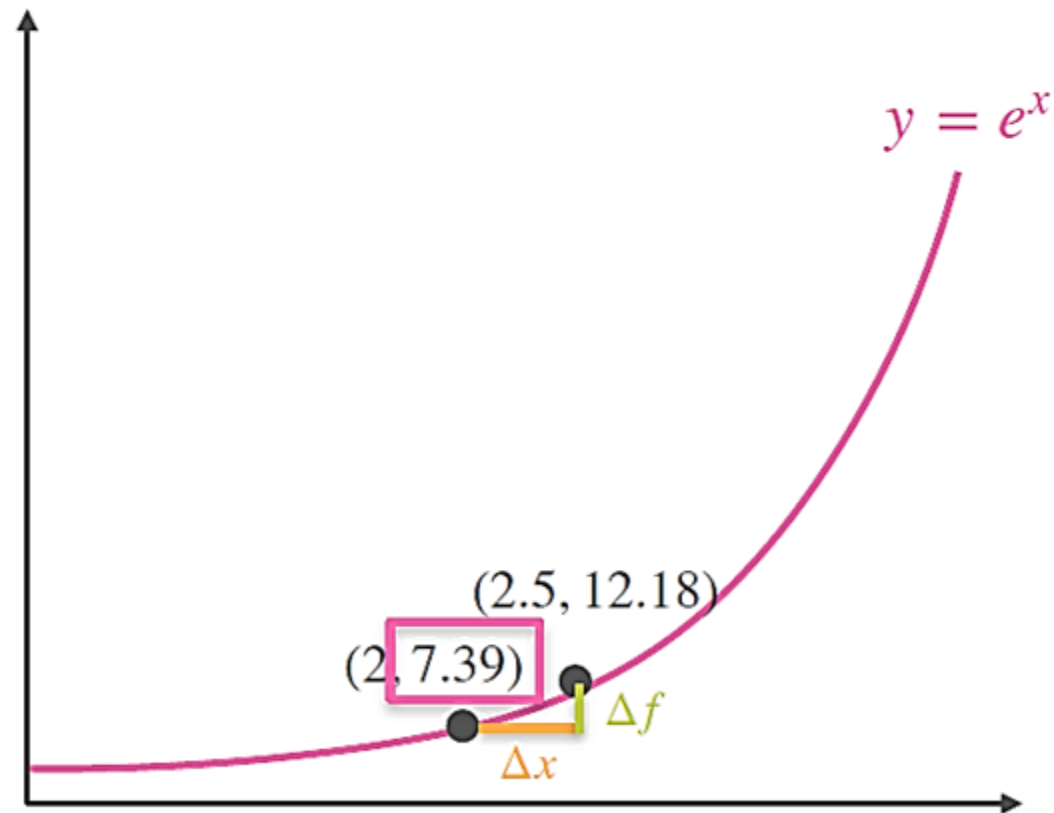
$$\text{Slope} = \frac{df}{dx} = \frac{e^{(x + \Delta x)} - e^x}{\Delta x}$$

if $x = 2$

Δx	1
Δf	12.7
Slope	12.7

$$df = e^{(2 + 1)} - e^2 = 20.09 - 7.39 = 12.7$$

$$\text{Slope} = \frac{df}{dx} = \frac{12.7}{1} = 12.7$$

Derivative of e^x 

In this case, $y = f(x) = e^x$, so $f'(x) = e^x$, thus:

$$\text{Slope} = \frac{df}{dx} = \frac{e^{(x + \Delta x)} - e^x}{\Delta x}$$

if $x = 2$

Δx	1	$1/2$	$1/4$	$1/8$	$1/16$	$1/1000$
Δf	12.7	4.79	2.10	0.98	0.48	0.007
Slope	12.7	9.59	8.39	7.87	7.62	7.39

e^2

What is Logarithm?

The power that raised to a certain base to get a given number.

$$x^y = z$$

Base

The number raised to a certain power.

Exponent

The number that tells how many times to multiply the base by itself.

Result

The result of x multiplied by itself y times.

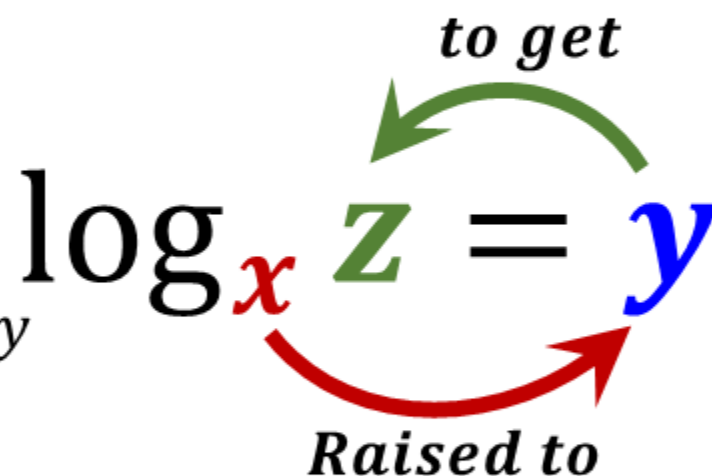
Logarithm reverses the exponential expressions:

Exponent form: $x^y = z$ *x to y equals z*

$$5^3 = 125$$

Logarithm form: $\log_x z = y$ *log base x of z equals y*

$$\log_5 125 = 3$$



Thus, the logarithm function is the inverse of the exponential function

$$f(x) = e^x \rightarrow f^{-1}(x) = \log_e(x) = \ln(x)$$

$$f(x) = ex \rightarrow f^{-1}(x) = \log_e(x) = \ln(x)$$

$$f(x) = ex$$

$$y = ex$$

$$x = ey$$

$$\ln(x) = \ln(ey)$$

$$\ln e = 1$$

$$\ln(x) = y$$

$$\ln(x) = \log_e(x)$$

$$y = \ln x$$

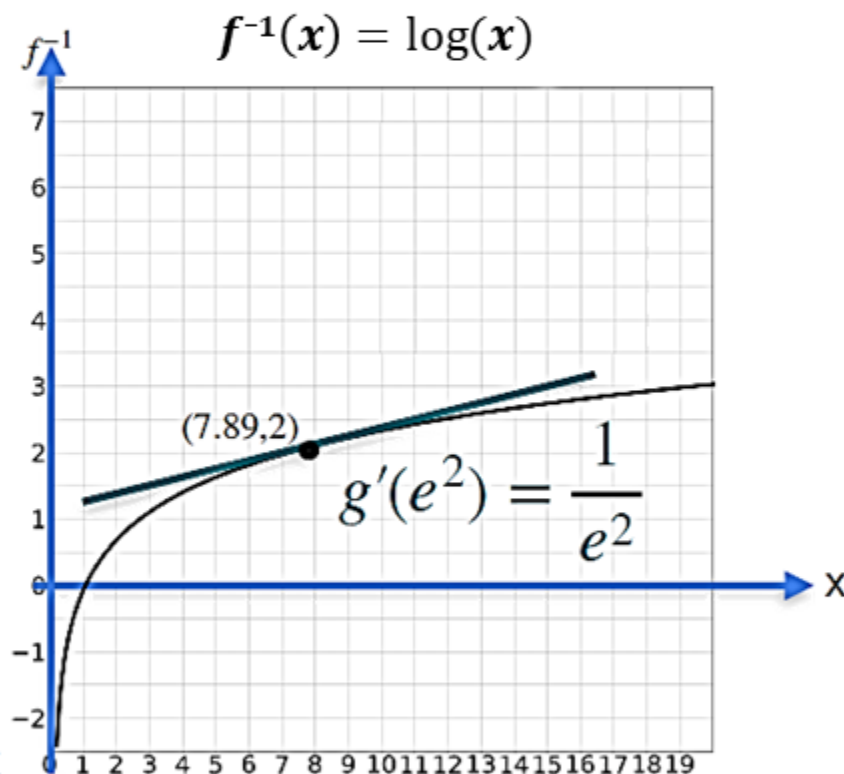
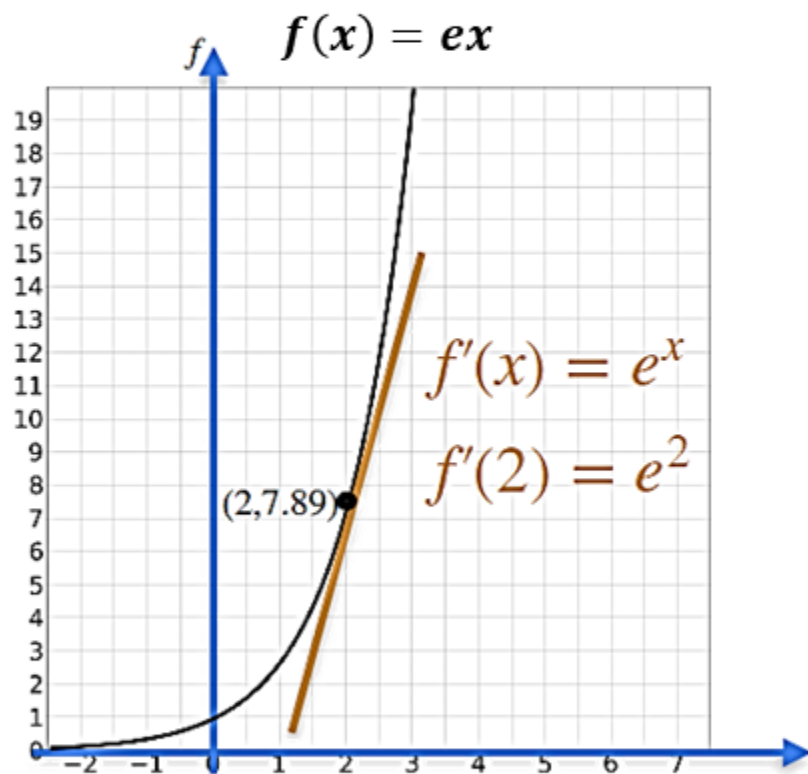
$$e^y = x$$

Example:

$$1.609 = \ln 5$$

$$e^{1.609} = 5$$

Derivatives of Logarithm



$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(x)} = \frac{1}{e^x}$$

$$\text{Since } \frac{d}{dx} f^{-1}(x) = \frac{1}{e^2}$$

$e^2 = 7.89$, value of x

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{x}$$

Since $f^{-1}(x) = \log(x)$

$$\therefore \frac{d}{dx} \log(x) = \frac{1}{x}$$

Properties of the derivative; Multiplication by Scalars

$$f(x) = c g(x)$$

Diagram illustrating the multiplication of a function by a scalar:

- Function** (red box) points to $f(x)$
- Scalar** (blue box) points to c
- Function** (green box) points to $g(x)$

$$f'(x) = c g'(x)$$

Diagram illustrating the derivative of a function multiplied by a scalar:

- Derivative of $f(x)$** (red box) points to $f'(x)$
- Scalar** (blue box) points to c
- Derivative of $g(x)$** (green box) points to $g'(x)$

Properties of the derivative; Multiplication by Scalars

Examples

$$f(x) = x^2, g(x) = 2x^2 \quad \text{And } f(x) = 4g(x)$$

$$x^2 = 4(2x^2)$$

Also,

$$f'(x) = 4g'(x)$$

$$2x = 4(4x)$$

Properties of the derivative; Sum Rule

$$f(x) = h(x) + g(x)$$

Function

Function

Function

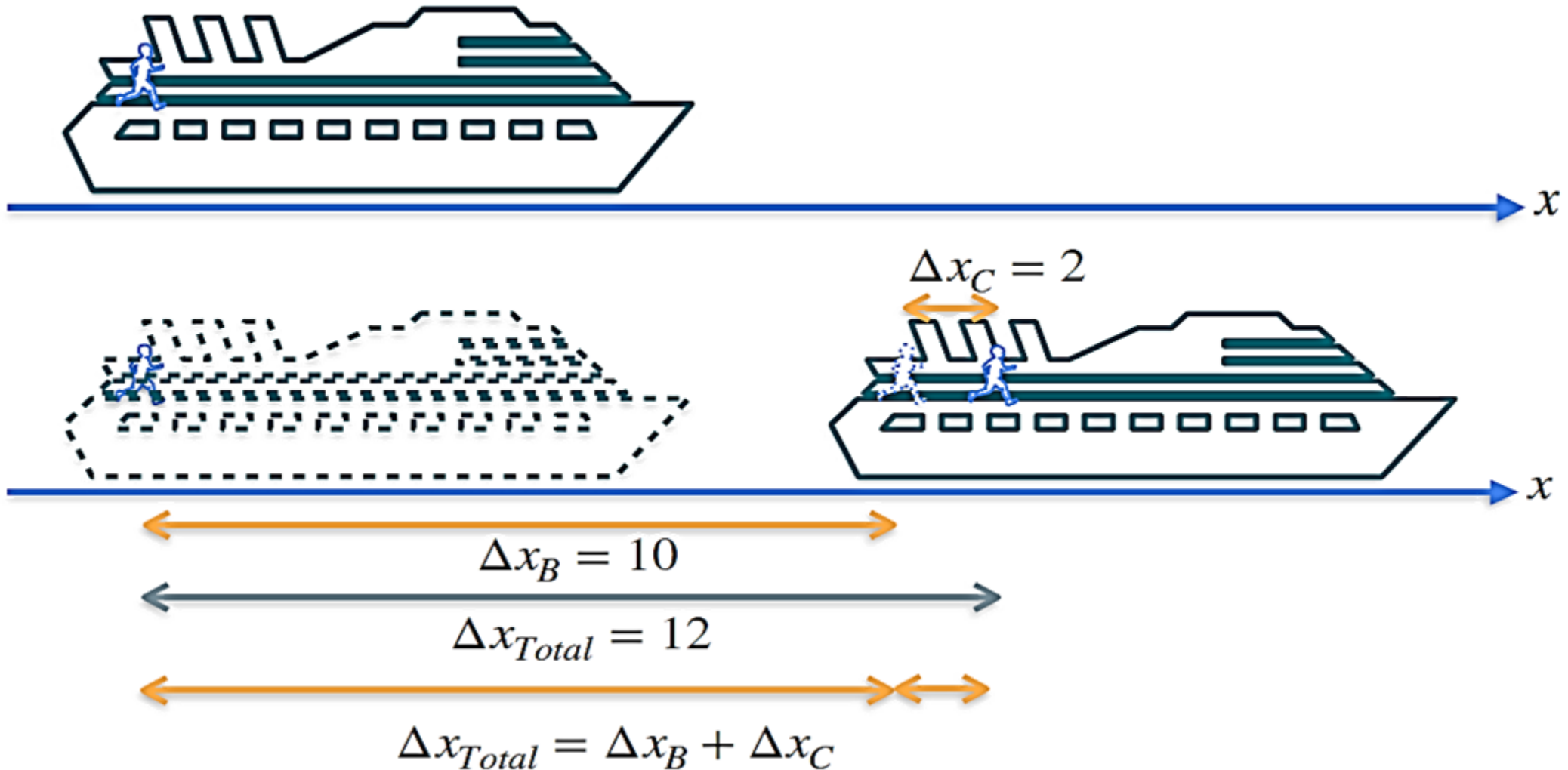
$$f'(x) = h'(x) + g'(x)$$

Derivative of
 $f(x)$

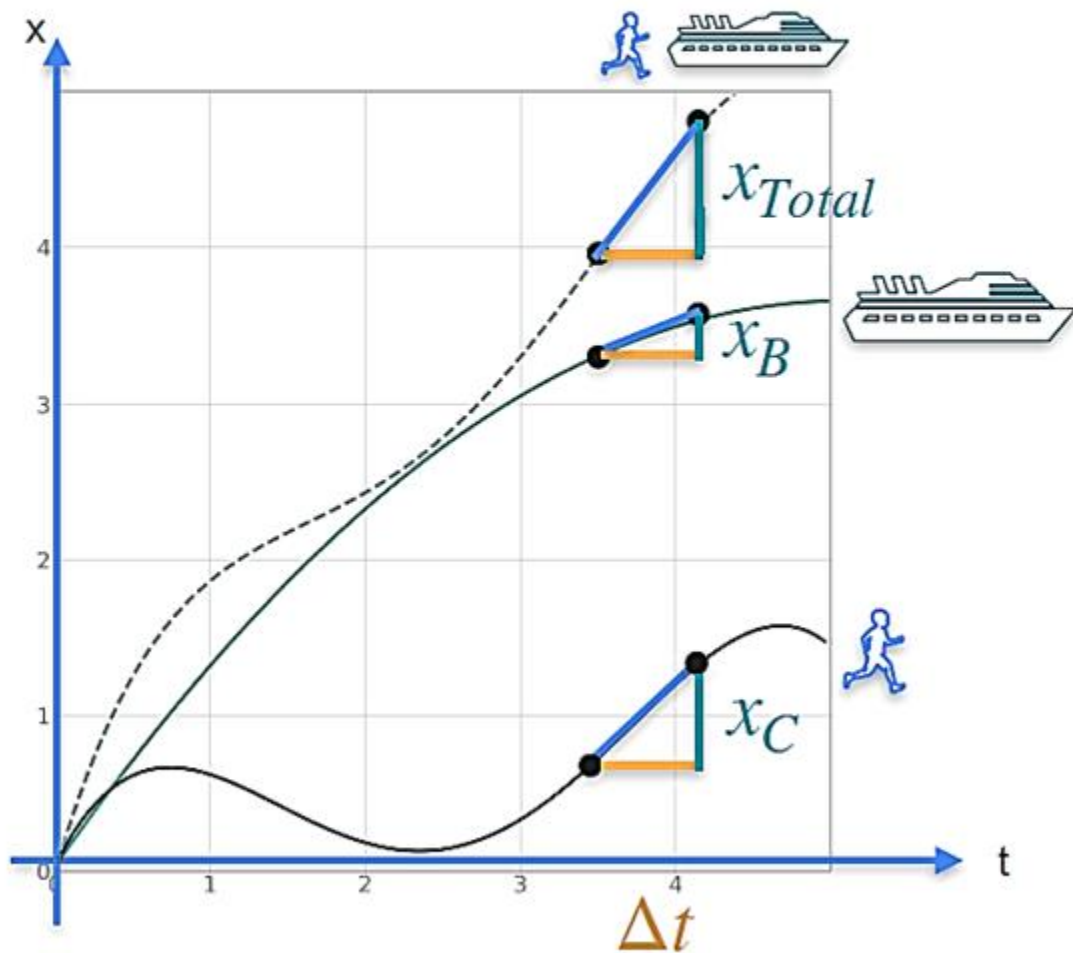
Derivative of
 $h(x)$

Derivative of
 $g(x)$

Properties of the derivative; Sum Rule



MIDOCEAN
UNIVERSITY
BY EVERYONE'S SMART UNIVERSITY

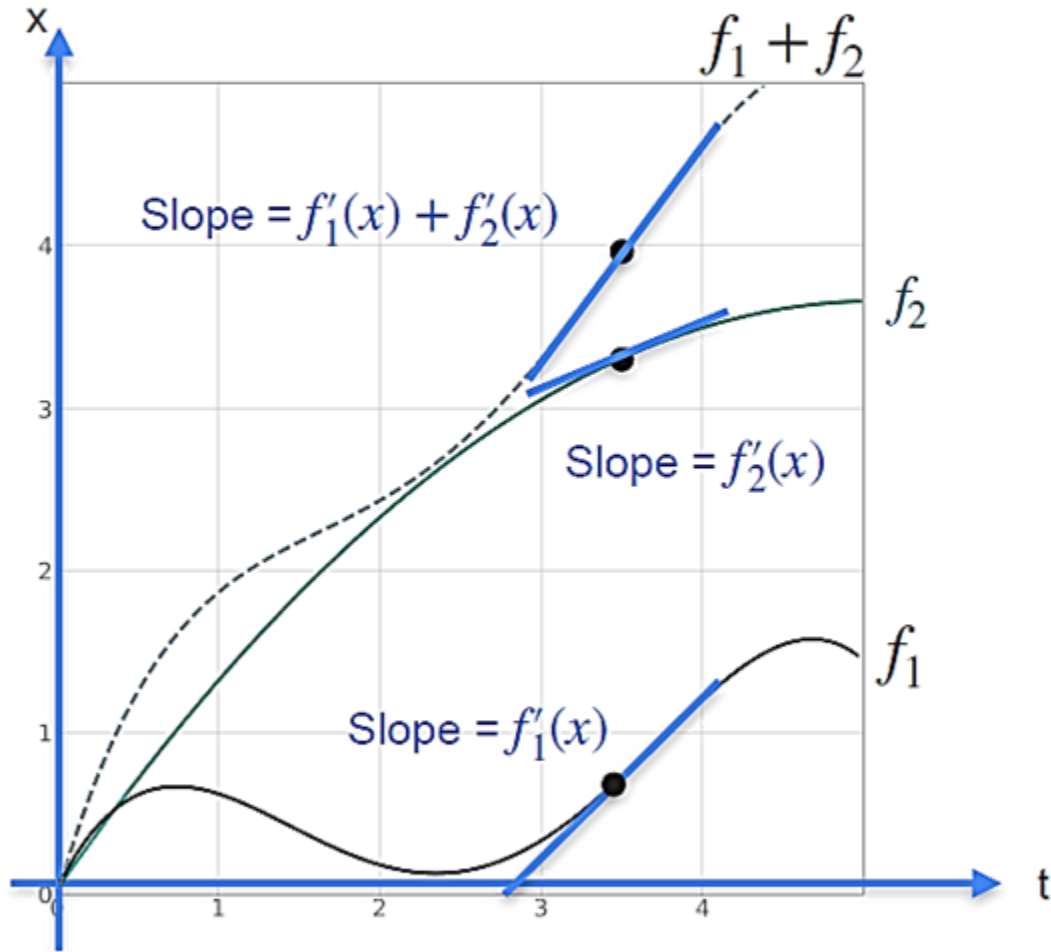


$$x_{Total} = xB + xC$$

$$\frac{\mathbf{x}_{Total}}{\Delta t} = \frac{\mathbf{x}_B}{\Delta t} + \frac{\mathbf{x}_C}{\Delta t}$$

$$v_{Total} = vB + vC$$

Properties of the derivative; Sum Rule



$$f_1 = f_2 + f_3$$



$$f'_1 = f'_2 + f'_3$$

Properties of the derivative; Product Rule

$$f(x) = h(x) \times g(x)$$

Function

Function

Function

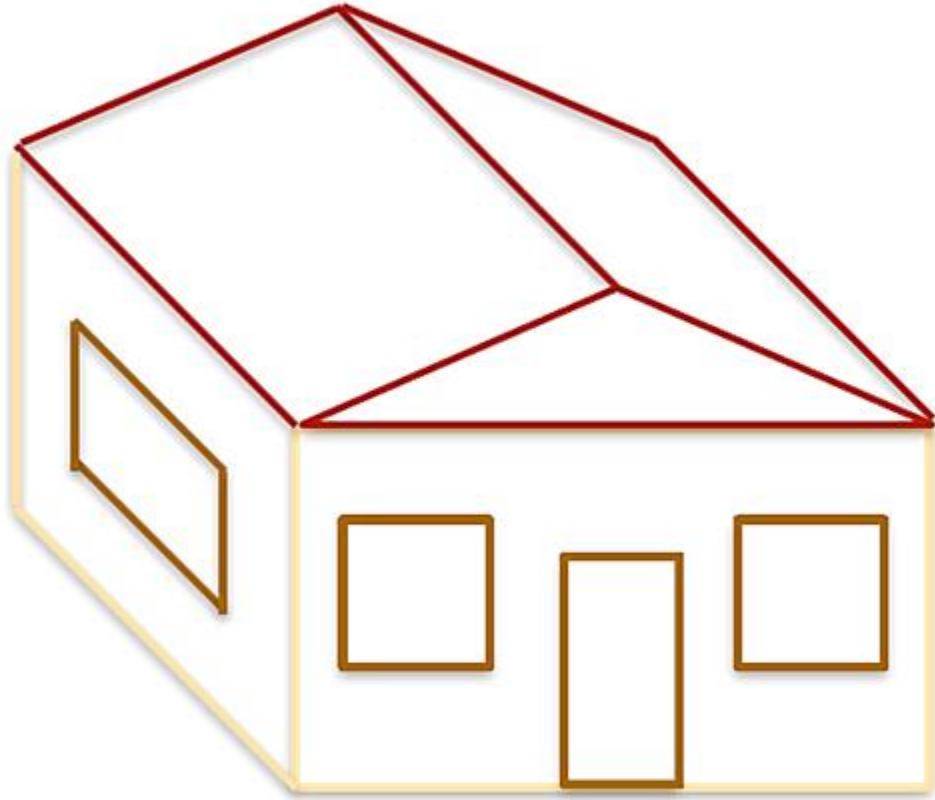
$$f'(x) = g(x)h'(x) + h(x)g'(x)$$

Derivative of
 $f(x)$

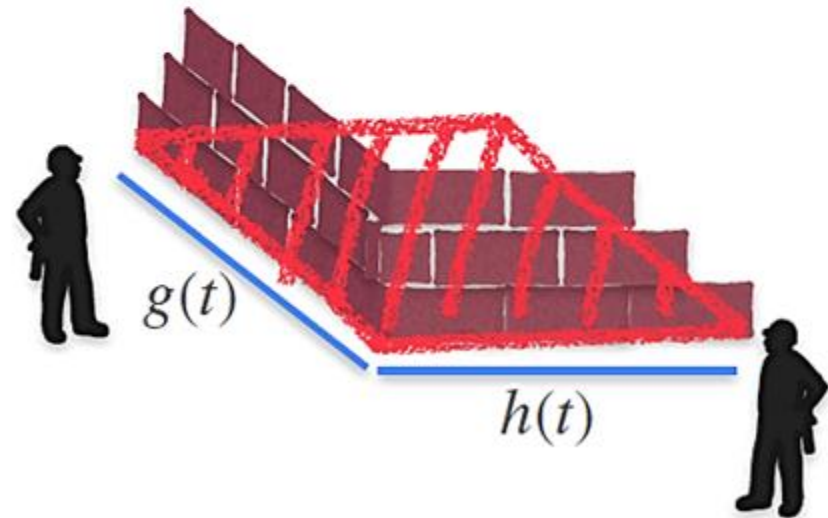
Derivative of
 $h(x)$

Derivative of
 $g(x)$

Properties of the derivative; Product Rule



$$f(t) = g(t) \times h(t)$$



Properties of the derivative; Chain Rule

$$\frac{d}{dx} g(h(x)) = \frac{dg}{dh} \times \frac{dh}{dx}$$

$$\frac{d}{dx} f(g(h(x))) = \frac{df}{dg} \times \frac{dg}{dh} \times \frac{dh}{dx}$$

Exercise

The following table lists the values of functions g and h , and of their derivatives, g' and h' , for $x = -2$

Let function $f(x)$ defined as $g(x) \times h(x)$

What is the value of $f'(-2)$?

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
-2	2	-1	3	4

Exercise

The following table lists the values of functions f and h and of their derivatives, f' and h' , for $x = -1$

Evaluate $[f(x) \times h(x)]$ at $x = -1$

x	$f(x)$	$h(x)$	$f'(x)$	$h'(x)$
-1	-3	-4	1	3

Exercise

The following table lists the values of functions f and h and of their derivatives, f' and h' , for $x = -1$

Evalute $[f(x) \times h(x)]$ at $x = -1$

x	$f(x)$	$h(x)$	$f'(x)$	$h'(x)$
-1	-3	-4	1	3

Exercise

Let f be a function such $f(-1) = 3$, and $f'(-1) = 5$

Let g be function $g(x) = \frac{1}{x}$.

Let h be a function defined as $h(x) = f(x) \times g(x)$, What is the value of $h'(-1)$?

Function	Derivatives
$f(x) = \sin(x)$	$f'(x) = \cos(x)$
$f(x) = \cos(x)$	$f'(x) = -\sin(x)$
$f^{-1}(x)$	$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(x)}$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \log(x)$	$\frac{d}{dx}\log(x) = \frac{1}{x}$
$f(x) = cg(x)$	$f'(x) = cg'(x)$
$f(x) = h(x) + g(x)$	$f'(x) = h'(x) + g'(x)$
$f(x) = h(x) \times g(x)$	$f'(x) = g(x)h'(x) + h(x)g'(x)$
$f(g(h(x)))$	$\frac{d}{dx}f(g(h(x))) = \frac{df}{dg} \times \frac{dg}{dh} \times \frac{dh}{dx}$

