

Digital Systems- ITC131

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Lecture 03: Boolean Algebra

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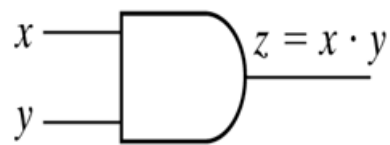
- Why study Boolean Algebra?
- Binary Logic and Gates
- Boolean Algebra
- Canonical and Standard Forms

Binary Logic and Logic Gates

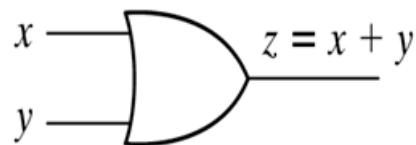
- Our objective is to learn how to design digital circuits.
- These circuits use binary systems and Deals with binary variables and with operations that assume logical meaning.
- Manipulation of binary information is done using circuits called *logic gates*.

Binary Logic and Logic Gates

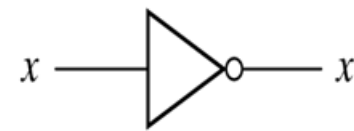
- **Logic gate** is a hardware that produce signals of binary when input logic requirements are satisfied.
- Basic **logical operators** are the logic functions AND, OR and NOT.



(a) Two-input AND gate



(b) Two-input OR gate



(c) NOT gate or inverter

- **Boolean Algebra** is a system of mathematics based on logic that has its own set of rules or laws which are used to:
 - define Boolean expressions.
 - reduce Boolean expressions.
 - analyse and simplify the digital (logic) circuits.
- The variables used in Boolean Algebra have only one of two possible values, a logic "0" and a logic "1"

- **Boolean Algebra** is made up of
 1. **Elements** - variables or constants with value 1 or 0.
 2. **Operators** - AND, OR and NOT etc.
 3. **Axioms and Theorems** - self-evident universally accepted truth.
- There are three ways to represent a Boolean Expression/Function.
 1. Algebraic Equation
 2. Truth Table
 3. Circuit Diagram

➤ **Boolean function:** expression consists of binary variables, constants (0/1), and logic operations.

It can be equal 1 or 0.

➤ The output of any function can be equal to 0 or 1.

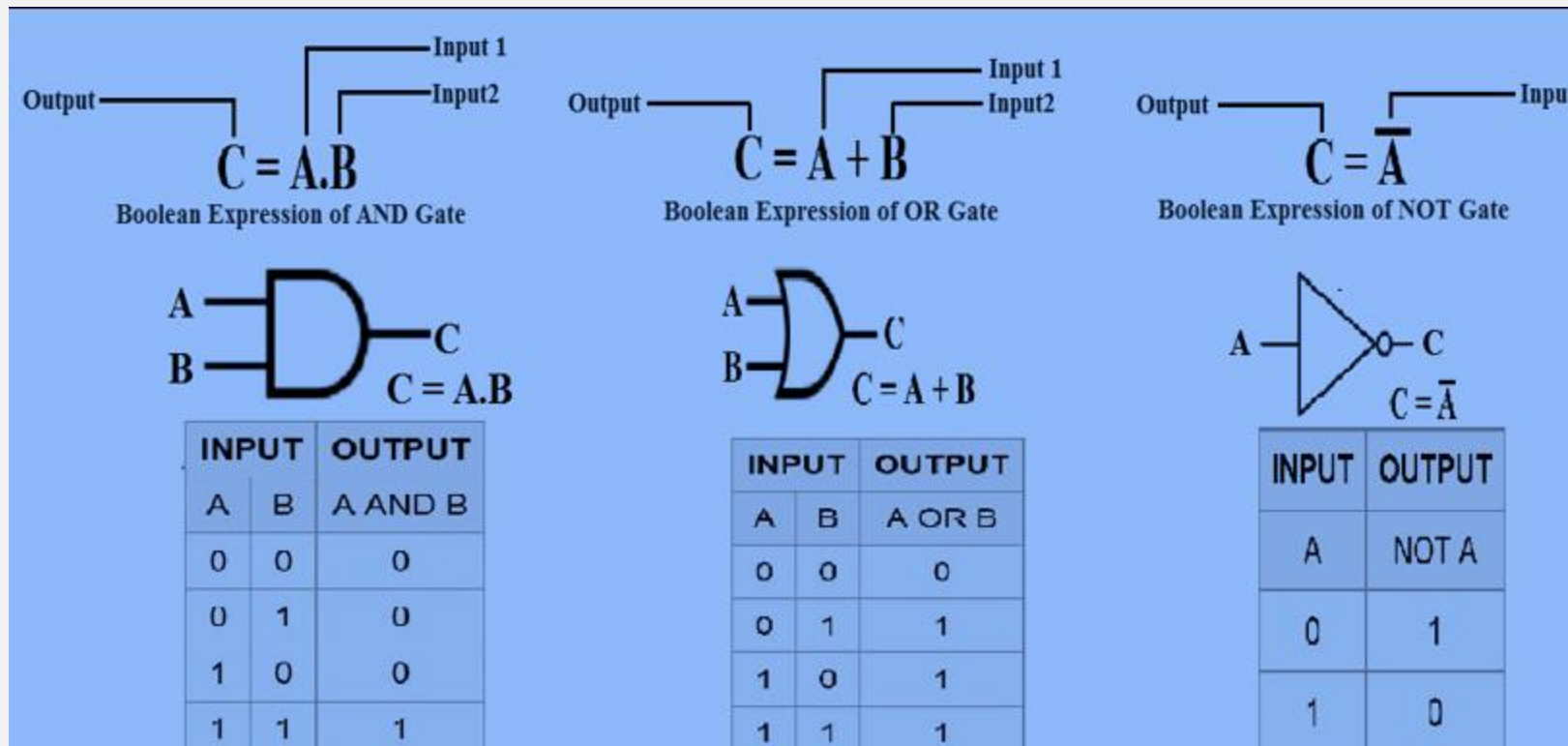
➤ **Example:** $F = x + y' z$;

➤ **Question:** When $F=1$ and $F=0$?

- It can be presented by Truth Table, where number of rows is equal to 2^n where n is number of variables.
- The combinations are binary numbers from 0 to 2^n-1 .
- **Boolean variable**: a variable x is a Boolean variable if value of $x \in B$
- **Literal**: a variable in complemented or uncomplemented forms.
- **Truth table** - a tabular listing of the values of a function for all possible combinations of values on its arguments

Boolean Algebra

Truth Table of Basic Gates



- In Boolean Algebra, We are going to focus on :
- Simplification of Boolean expression
- Complementing functions
- Representing Boolean functions (Canonical and Standard Forms)

Boolean Function Evaluation

- The **operator precedence** for evaluating Boolean Expression is
 1. Parentheses
 2. NOT (Complement)
 3. AND
 4. OR

Expression Simplification

- By manipulating a Boolean expression according to Boolean algebraic rules, it is often possible to obtain a simpler expression for the same function. This simpler expression **reduces both the number of gates in the circuit and the numbers of inputs to the gates.**
- To see how this is done, we must first study the basic rules of Boolean algebra.

Boolean Algebra

Basic identities of Boolean algebra

• T1 : Properties of 0

- (a) $0 + X = X$
- (b) $0.X = 0$

• T2 : Properties of 1

- (a) $1 + X = 1$
- (b) $1.X = X$

$$0 + X = X$$

0	X	R
0	0	0
0	1	1

$$1 + X = 1$$

1	X	R
1	0	1
1	1	1

$$0 . X = 0$$

0	X	R
0	0	0
0	1	0

$$1 . X = X$$

1	X	R
1	0	0
1	1	1

Boolean Algebra

Basic identities of Boolean algebra

• T3 : Commutative Law

- (a) $X + Y = Y + X$
- (b) $X.Y = Y.X$

X	Y	X+Y	Y+X
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

X	Y	X.Y	Y.X
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

Basic identities of Boolean algebra

• T4 : Associate Law

• (a) $(X + Y) + Z = X + (Y + Z)$

X	Y	Z	Y . Z	X . Y	X . (Y . Z)	(X . Y) . Z
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	1	0	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	0	1	0	0
1	1	1	1	1	1	1

(b) $(X.Y).Z = X.(Y.Z)$

X	Y	Z	Y + Z	X + Y	X + (Y + Z)	(X + Y) + Z
0	0	0	0	0	0	0
0	0	1	1	0	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Basic identities of Boolean algebra

• T5 : Distributive Law

• (a) $X(Y + Z) = X.Y + X.Z$

(b) $X + (Y.Z) = (X + Y)(X + Z)$

X	Y	Z	Y+Z	X.Y	X.Z	X.(Y+Z)	X.Y+X.Z
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

X	Y	Z	Y.Z	X+Y	X+Z	X+Y.Z	(X+Y).(X+Z)
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Basic identities of Boolean algebra

• T6 : Idempotence Law

• (a) $X + X = X$

(b) $X.X = X$

X	X	R
0	0	0
1	1	1

X	X	R
0	0	0
1	1	1

Boolean Algebra

Basic identities of Boolean algebra

• T7 : Absorption (Redundance) Law

• (a) $X + X.Y = X$

(b) $X .(X + Y) = X$

X	Y	X.Y	X+X.Y
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

X	Y	X + Y	X.(X+Y)
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

• $X + X.Y = X.(1+Y)$

• $= X.(1)$

• $= X$

$X.(X+Y)$

$X.X + X.Y =$

$X + X.Y = X$

Basic identities of Boolean algebra

• T8 : Complementary Law

• (a) $X + X' = 1$

(b) $X \cdot X' = 0$

X	X'	R
0	1	1
1	0	1

X	X'	R
0	1	0
1	0	0

Basic identities of Boolean algebra

- T9 : Involution

- (a) $x'' = x$

- T10 : De Morgan's Theorem

- (a) $(X+Y)' = X' \cdot Y'$

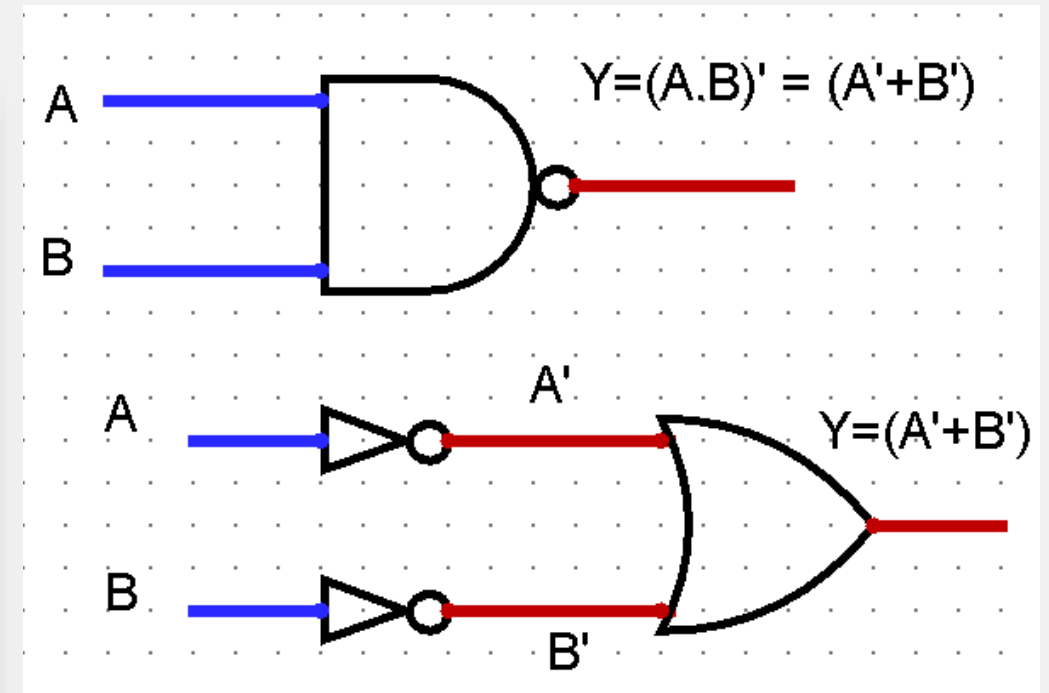
- (b) $(X \cdot Y)' = X' + Y'$

Boolean Algebra

De Morgan's Theorem

- $(A \cdot B)' = (A' + B')$

A	B	$\overline{AB}$	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	1	1	1	1
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

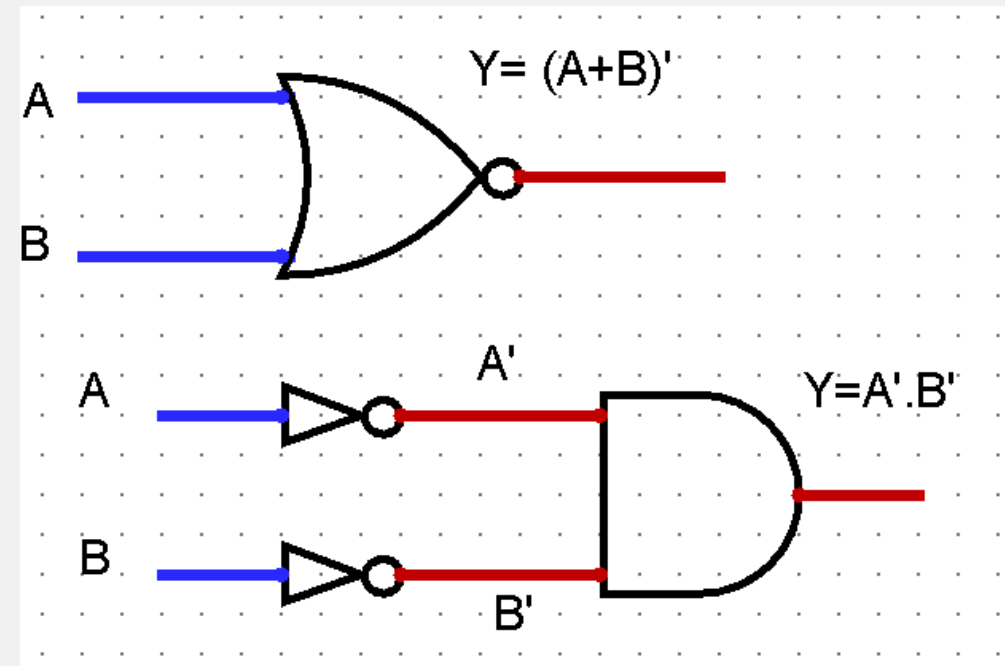


Boolean Algebra

De Morgan's Theorem

$$(A + B)' = A' \cdot B'$$

A	B	$\overline{A+B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cdot \overline{B}$
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	0



- When a Boolean expression is implemented with logic gates, each term requires a gate and each **variable****literals** designates an input to gate.
- Simplify to contain the smallest number of literals (complemented and un-complemented variables) and **terms**

has three
terms and
eight literals

$$F = X'YZ + X'YZ' + XZ$$

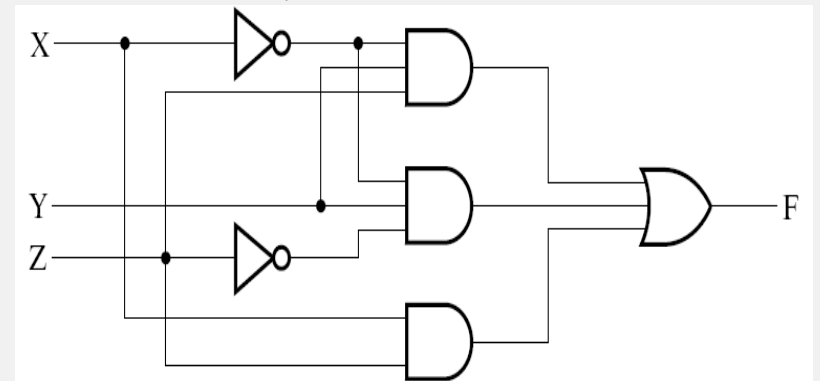
$$F = X'Y(Z + Z') + XZ$$

$$F = X'Y.1 + XZ$$

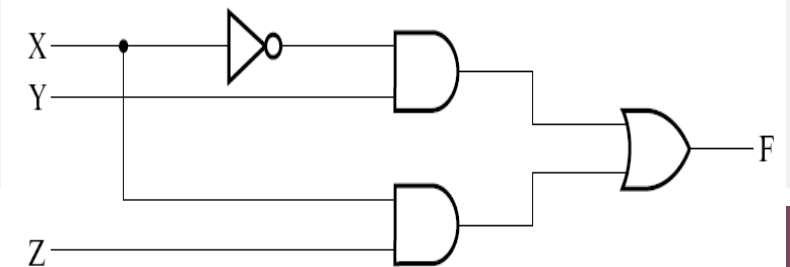
has two
terms and
four literals

$$F = X'Y + XZ$$

It is highly desirable to find the **simplest circuit** implementation with the **smallest number of gates or wires**.



(a) $F = \bar{X}YZ + \bar{X}YZ' + XZ$



(b) $F = \bar{X}Y + XZ$

Example 1

For the given truth table, devise the Boolean equation.

Solution

$$F = X'Y'Z + XY'Z' + XY'Z + XYZ' + XYZ$$

Simplify:

$$F = X'Y'Z + XY'(Z' + Z) + XY(Z' + Z)$$

$$F = X'Y'Z + XY' + XY$$

$$F = X'Y'Z + X(Y' + Y)$$

$$F = X'Y'Z + X$$

Truth Table

X Y Z	F
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Complementing Functions

- The complement (inverse) a function $F = F'$ is obtained from interchanging 0's to 1's in the value of F in the truth table.
- A Boolean equation for F' can be derived algebraically by using *DeMorgan's Theorem*:
 - *interchange AND and OR operators*
 - *complement each constant value and literal*

• Examples:

$$F = x'yz' + xy'z'$$

$$F' = (x + y' + z)(x' + y + z)$$

$$G = (a' + bc)d' + e$$

$$G' = (a(b' + c') + d)e'$$

- Exercises:

- - Simplify the following Boolean expressions to a minimum number of literals:

- a. $F(x,y) = xy + xy'$

$$F = x(y + y') = x.1 = x$$

- b. $F(x,y) = (x + y)(x + y')$

$$F = xx + xy' + yx + yy' = x + xy' + xy + 0 = x(1 + y' + y) = x.1 = x$$

- c. $F(w,x,y,z) = xy + x(wz + wz')$

$$F(w,x,y,z) = xy + xw(z+z') = xy + xw = x(y+w)$$

- d. $F(A,B,C,D) = (BC' + A'D)(AB' + CD')$

$$\begin{aligned} F(A,B,C,D) &= AB'BC' + AB'A'D + CD'BC' + CD'A'D \\ &= 0 \end{aligned}$$

1. Reduce the following Boolean expressions to the indicated number of literals:

$$F(x,y,z) = x'z' + xyz + xz'$$

to three literals

$$F = z' (x+x') + xyz$$

$$F = (z + z')(z' + xy)$$

$$F = xy + z'$$

For the Boolean function

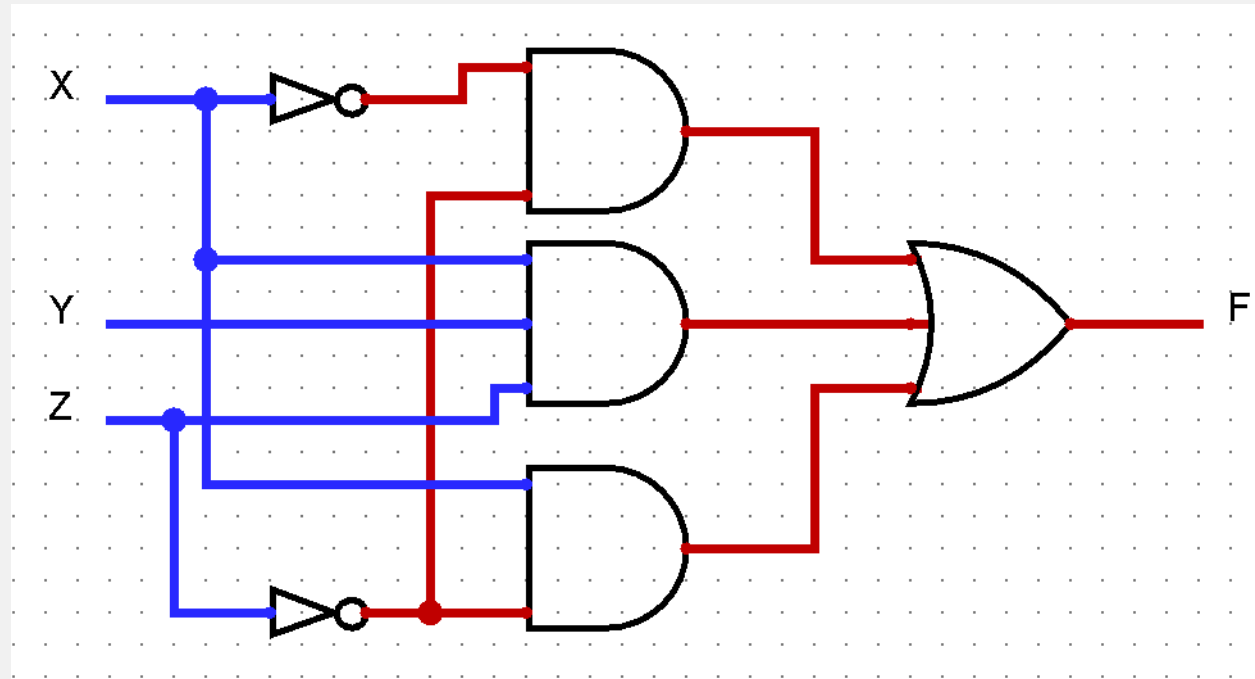
$$F(x,y,z) = x'z' + xyz + xz'$$

1. Obtain the truth table of F.
2. Draw the logic diagram for the original Boolean expression.
3. Use Boolean algebra to simplify the function to a minimum number of literals,
4. Draw the logic diagram for the simplified Boolean expression.

- Obtain the truth table of F.

X	Y	Z	X'	Z'	X'Z'	XYZ	XZ'	F
0	0	0	1	1	1	0	0	1
0	0	1	1	0	0	0	0	0
0	1	0	1	1	1	0	0	1
0	1	1	1	0	0	0	0	0
1	0	0	0	1	0	0	1	1
1	0	1	0	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	1	1	0	0	0	1	0	1

- Draw the logic diagram for the original Boolean expression.
- $F(x,y,z) = x'z' + xyz + xz'$



- Use Boolean algebra to simplify the function to a minimum number of literals,

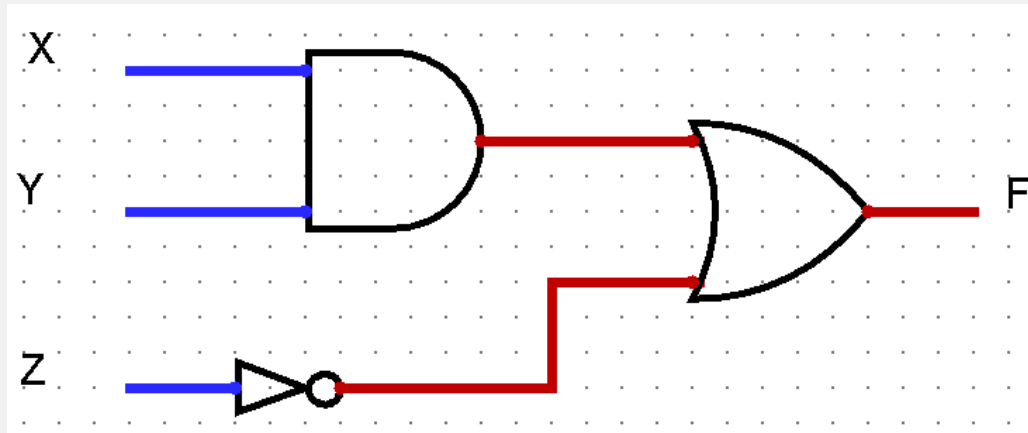
$$F(x,y,z) = x'z' + xyz + xz'$$

$$F = z' (x+x') + xyz$$

$$F = (z + z')(z' + xy)$$

$$F = xy + z'$$

- Draw the logic diagram for the simplified Boolean expression



X	Y	Z	XY	Z'	F
0	0	0	0	1	1
0	0	1	0	0	0
0	1	0	0	1	1
0	1	1	0	0	0
1	0	0	0	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	1	0	1

Reduce the following Boolean expressions to the indicated number of literals:

$$F(w,x,y,z) = w'x (z' + y'z) + x(w + w'y z) \quad \text{to one literal}$$

$$F = x(w'z' + w'y'z + w + w'yz)$$

$$F = x(w'z' + w + w'z(y + y'))$$

$$F = x(w + w'(z' + z))$$

$$F = x(w + w')$$

$$F = x$$

Reduce the following Boolean expressions to the indicated number of literals:

$$F(w,x,y,z) = wxy'z + w'xz + wxyz$$

to two literal

$$\begin{aligned} F &= xz(wy' + w' + wy) \\ &= xz(w(y + y') + w') \\ &= xz(w + w') \\ &= xz \end{aligned}$$

End of Lecture 3