

Mathematics

Lecture 6

MAT 122
Determinants

- Definition of the determinants
- Determinant for 2×2 matrix
- Determinant for 3×3 matrix – First Way
- Determinant for 3×3 matrix – Second Way

Definition

The **determinant** of a 2×2 matrix A is denoted $|A|$ and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Observe that the determinant of a 2×2 matrix is given by *the different of the products of the two diagonals* of the matrix.

The notation **det**(A) is also used for the determinant of A .

Let $A = (a_{ij})_n$ be a square matrix of order n . The **determinant** of A (written $\det(A)$ or $|A|$) is defined by:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{1j_1} a_{2j_2} \dots a_{nj_n}.$$

Note that Matrix is the singular form, matrices is the plural form!

- A matrix is an array of numbers that are arranged in rows and columns.
- A matrix is “square” if it has the same number of rows as columns.
- We will consider only 2x2 and 3x3 square matrices

$$\begin{bmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{bmatrix}$$

Note the difference in the matrix and the determinant of the matrix!

- Every square matrix has a determinant.
- The determinant of a matrix is a number.
- We will consider the determinants only of 2x2 and 3x3 matrices.

$$\begin{vmatrix} 1 & 3 \\ -\frac{1}{2} & 0 \end{vmatrix}$$

$$\begin{vmatrix} -3 & 8 & \frac{1}{4} \\ 2 & 0 & -\frac{3}{4} \\ 4 & 180 & 11 \end{vmatrix}$$

Determinants of 2×2 Matrix

Assuming A is an arbitrary 2×2 matrix A, where the elements are given by:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

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Determinants of 2*2 Matrix

$$\begin{vmatrix} 3 & 2 \\ -5 & 4 \end{vmatrix}$$

Notice the different symbol:
the straight lines tell you to
find the determinant!!

$$\begin{vmatrix} 3 & 2 \\ -5 & 4 \end{vmatrix}$$

$$= (3 * 4) - (-5 * 2)$$

$$= 12 - (-10)$$

$$= 22$$

Determinants of 3×3 Matrix – First Way

Determinants of 3*3 Matrix – First Way

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} & a_{11} & a_{12} \\ a_{21} & \cancel{a_{22}} & \cancel{a_{23}} & a_{21} & a_{22} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} & \cancel{a_{31}} & \cancel{a_{32}} \end{bmatrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

$$\begin{vmatrix} 2 & 0 & 3 \\ 1 & -2 & 5 \\ -1 & 4 & 2 \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 1 & -2 \\ -1 & 4 \end{vmatrix}$$

$$\begin{aligned} &= [(2)(-2)(2) + (0)(5)(-1) + (3)(1)(4)] \\ &\quad - [(3)(-2)(-1) + (2)(5)(4) + (0)(1)(2)] \\ &= [-8 + 0 + 12] - [6 + 40 + 0] \\ &= 4 - 6 - 40 = -42 \end{aligned}$$

Determinants of 3×3 Matrix – Second Way

Determinants of 3*3 Matrix – Second Way

The determinant of a 3×3 matrix is a little more tricky and is found as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants of 3*3 Matrix – Second Way

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$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

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Determinants of 3*3 Matrix – Second Way

The determinant of a 3×3 matrix is a little more tricky and is found as follows (for this case assume A is an arbitrary 3×3 matrix A, where the elements are given below).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

then the determinant of a this matrix is as follows:

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Consider an example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Its determinant can be obtained by:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 6 \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} + 9 \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 3(-3) - 6(-6) + 9(-3) = 0$$

The following properties are true for determinants of any order.

1. If every element of a row (column) is zero,

e.g., $\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} = 1 \times 0 - 2 \times 0 = 0$, then $|A| = 0$.

2. $|A^T| = |A|$  determinant of a matrix
= that of its transpose

3. $|AB| = |A||B|$

Let's Practice

Find the determinants for the following matrix

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

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$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$

Evaluate the Determinant

$$\Delta = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$\begin{aligned}\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} &= 1(2) - 2(4) \\ &= 2 - 8 \\ &= -6\end{aligned}$$

Evaluate the determinant :

$$\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$$

$$\begin{aligned}\text{Solution : } & \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} \\ &= 4 \times 5 - 2 \times (-3) \\ &= 20 + 6 = 26\end{aligned}$$

Evaluate the determinant of the following matrix A .

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= 1 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix}$$

$$= [(0 \times 1) - (1 \times 2)] - 2[(3 \times 1) - (1 \times 4)] - [(3 \times 2) - (0 \times 4)]$$

$$= -2 + 2 - 6$$

$$= -6$$

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= -3 \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= -3[(2 \times 1) - (-1 \times 2)] + 0[(1 \times 1) - (-1 \times 4)] - 1[(1 \times 2) - (2 \times 4)]$$

$$= -12 + 0 + 6 = -6$$

Evaluate the Determinant

$$\Delta = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - (-4) \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$
$$= 3(1 + 6) + 4(1 + 4) + 5(3 - 2)$$
$$= 3(7) + 4(5) + 5(1)$$
$$= 21 + 20 + 5$$
$$= 46$$

Evaluate the determinant :

$$\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3 \begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} + (-5) \begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

$$= 2(1 + 8) - 3(7 - 6) - 5(28 + 3)$$

$$= 18 - 3 - 155$$

$$= -140$$

Evaluate the following determinant

$$\begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix}$$

$$= 5(18+10) + 1(12-25)+4(-4 -15)$$

$$= 140 -13 -76 =140 - 89$$

$$= 51$$

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