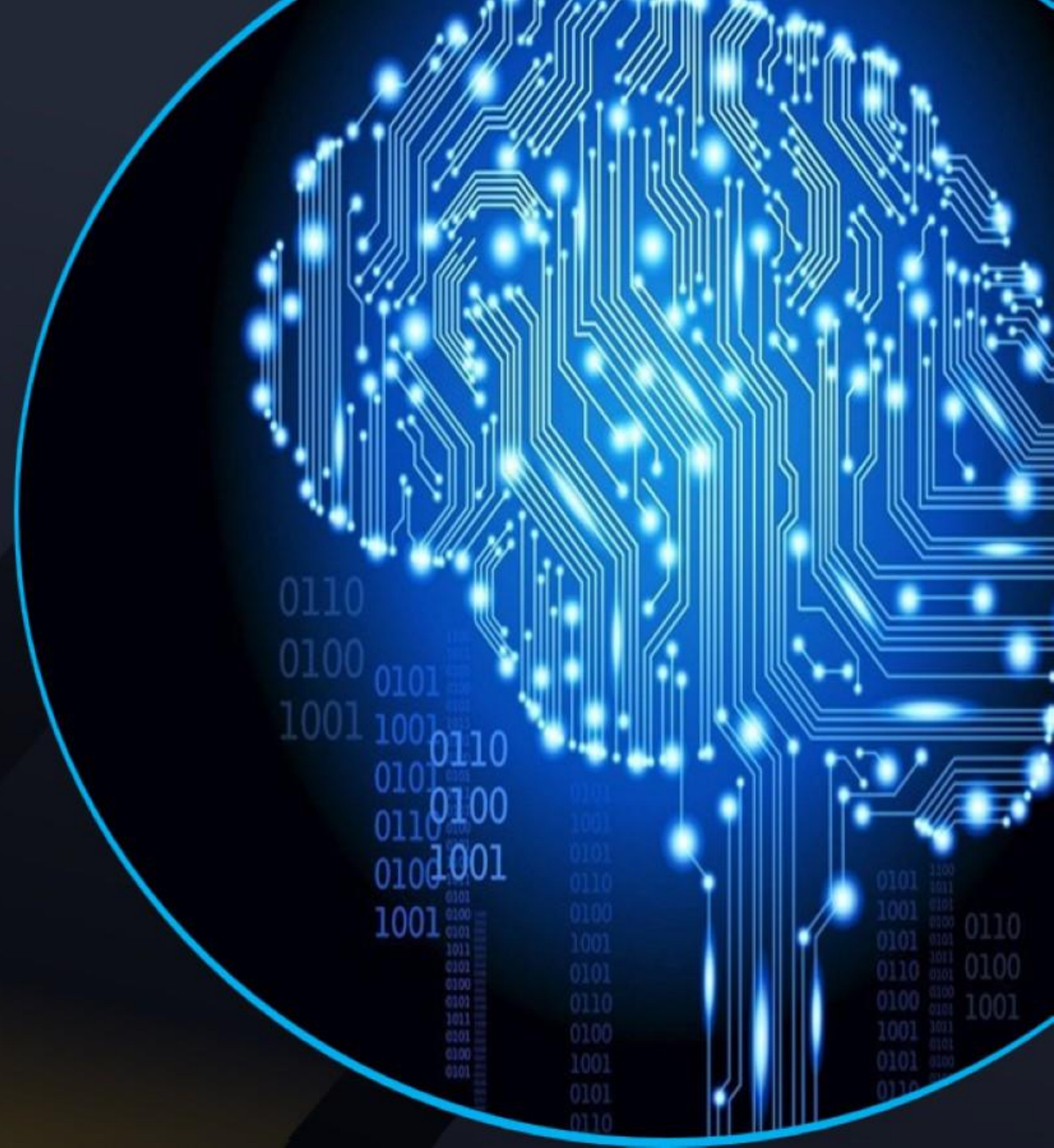


Discrete Mathematics

Dr. Shahira Ahmed

IT241

Lecture 4





Discrete Mathematics

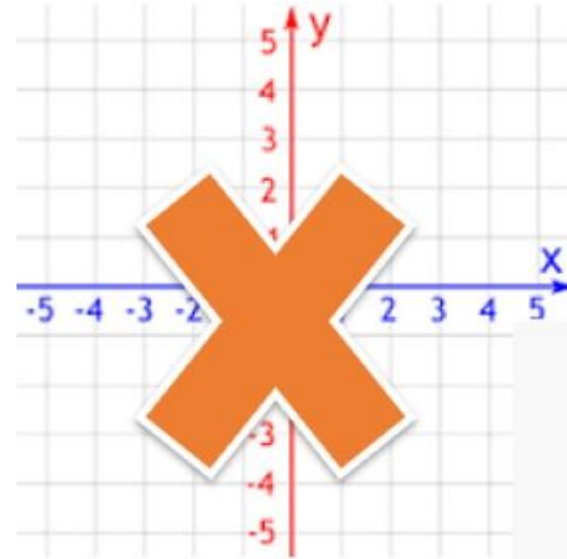
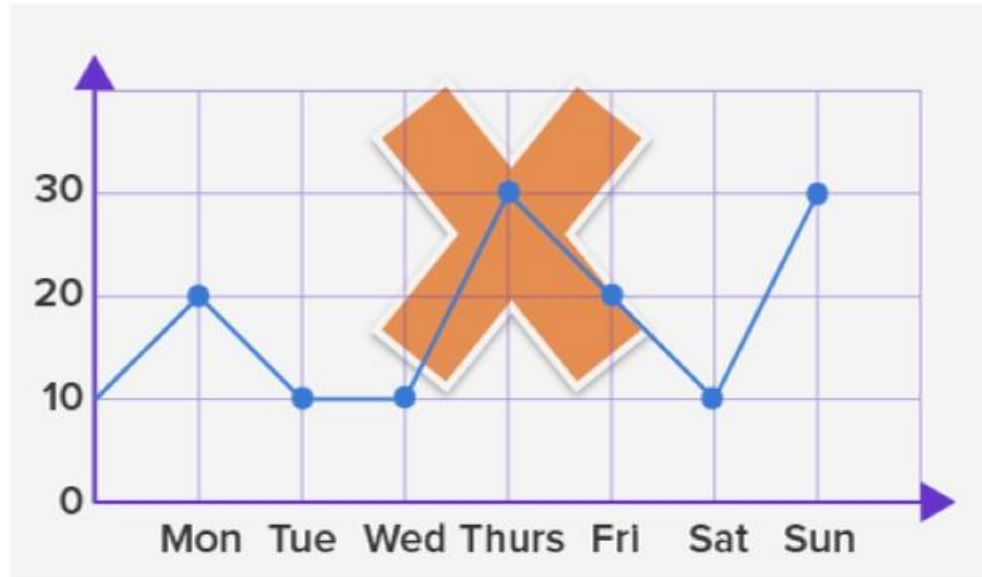
Lecture 4

Graph Theory



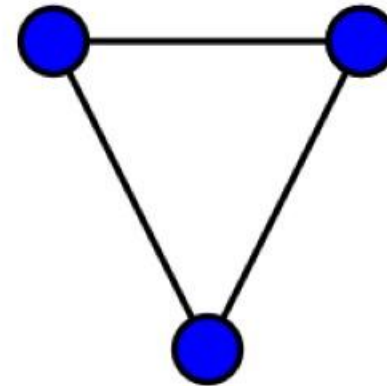
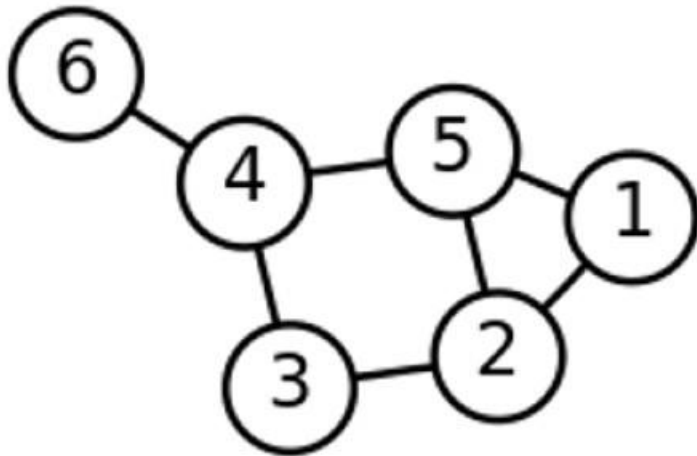
- Graph Definition
- Edge Type
- Graph Type
- Multiplicity
- Degrees
- Neighborhood
- Graph Representation

Not a Graph in discrete math



Not a Graph in discrete math

- Graph in discrete math



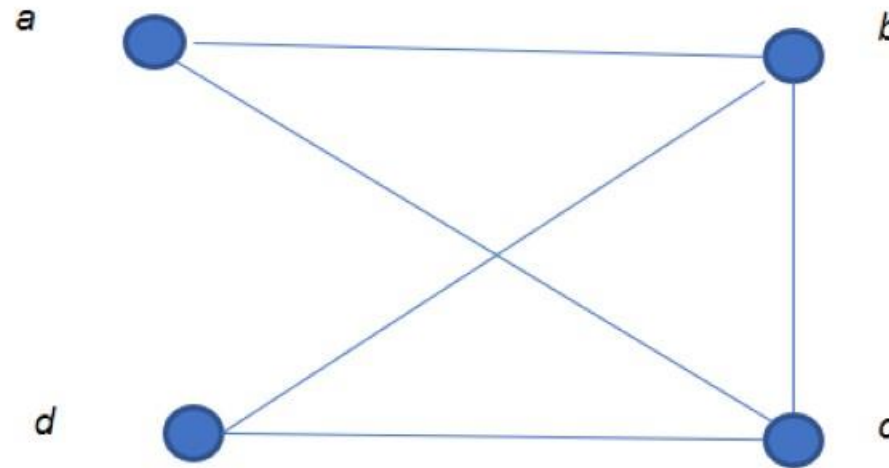
Representation: Graph $G = (V, E)$ consists set of vertices denoted by V , or by $V(G)$ and set of edges E , or $E(G)$.

How many Vertex?

How many Edge?

Solution

- four vertices
- five edges.



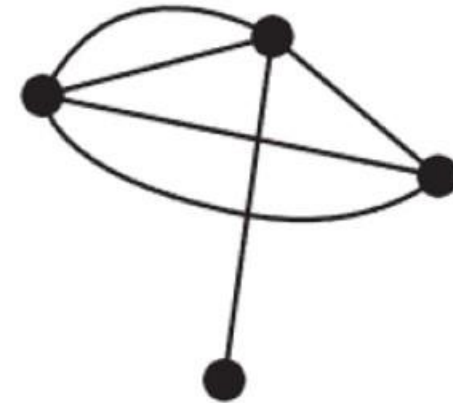
Directed: Ordered pair of vertices. Represented as (u, v) directed from vertex u to v .



Undirected: Unordered pair of vertices. Represented as $\{u, v\}$. Disregards any sense of direction and treats both end vertices interchangeably.

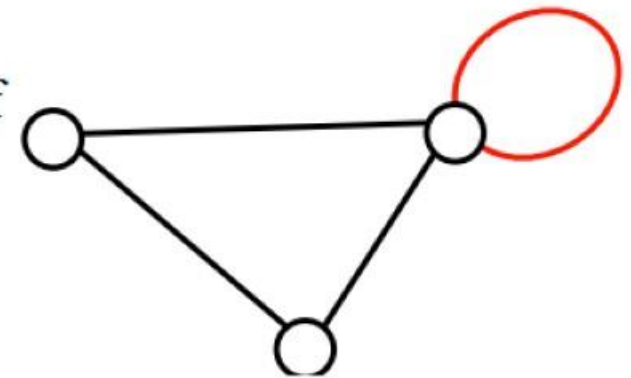


Multiple Edges: Two or more edges joining the same pair of vertices.

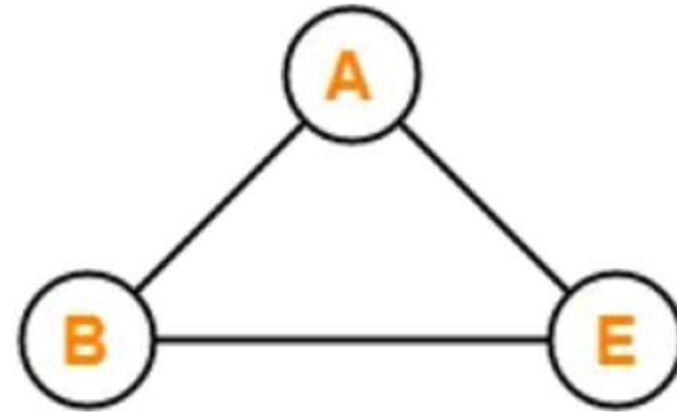


Loop: An edge that connects a vertex to itself

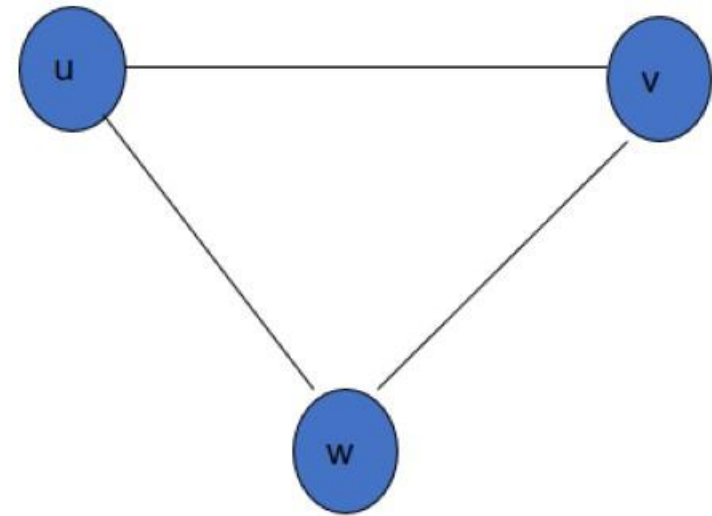
a loop (also called a self-loop) is an edge that connects a vertex to itself



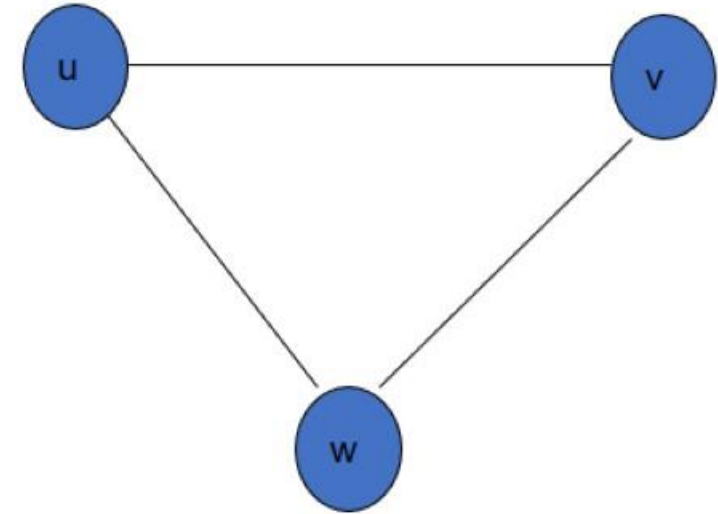
- A **Simple** graph have
 - no self loops
 - no multiple edges.



Simple (Undirected) Graph: consists of V , a nonempty set of vertices, and E , a set of **unordered pairs** of distinct elements of V called edges (**undirected**)

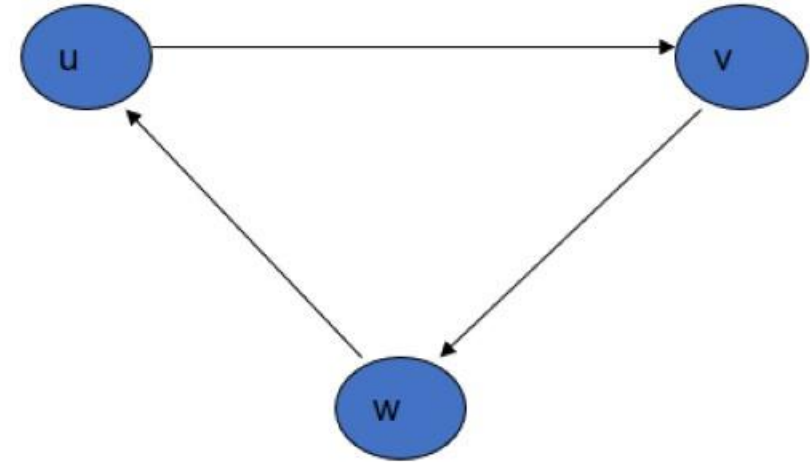


Represent graph in the opposite figure

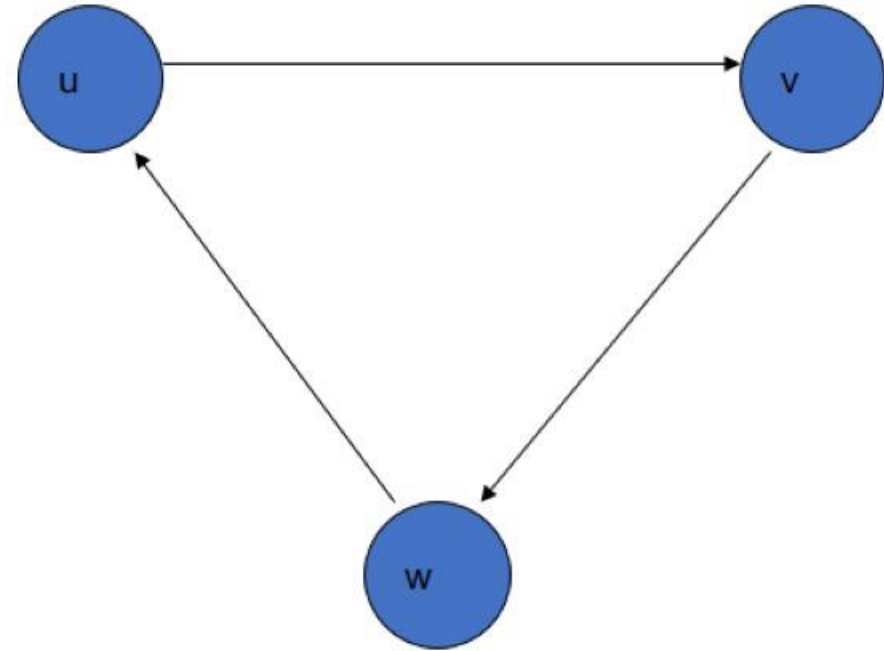


$$G(V, E), V = \{u, v, w\}, E = \{\{u, v\}, \{v, w\}, \{u, w\}\}$$

Simple Directed Graph: $G(V, E)$, set of vertices V , and set of Edges E , that are **ordered pair** of elements of V (directed edges)

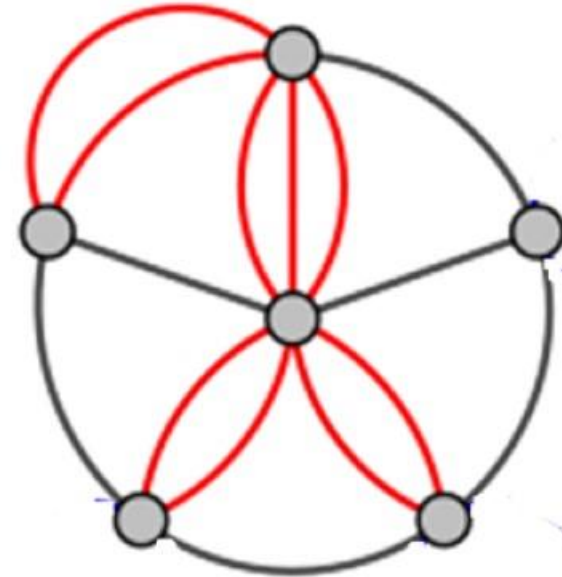


Represent Graph?



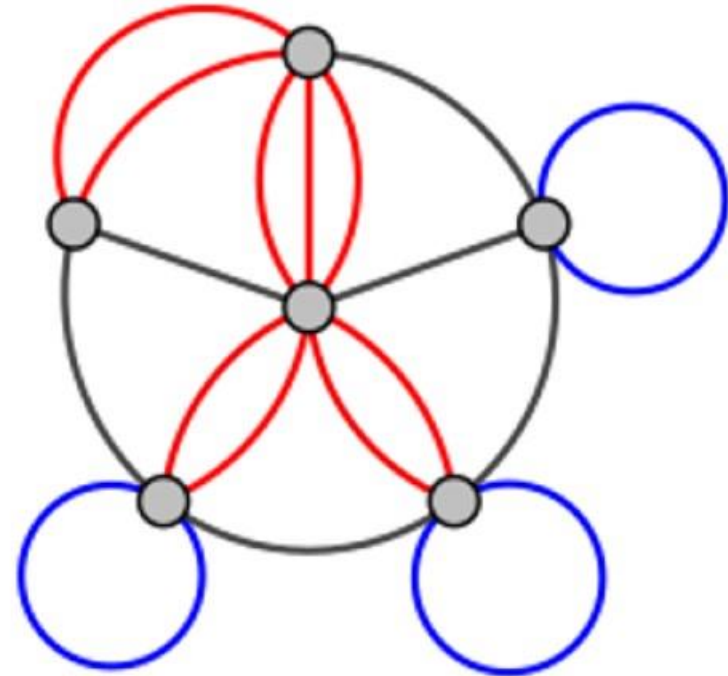
$$G(V, E), V = \{u, v, w\}, E = \{(u, v), (v, w), (w, u)\}$$

Multigraph: have **multiple** (parallel edges)



Pseudograph:

- **Loops** allowed in such a graph
- **Multiple** edges.



- A null graph is a graph in which there are **no edges** between its vertices.
- Edge set is **empty**



- A trivial graph is the graph which has only one vertex.

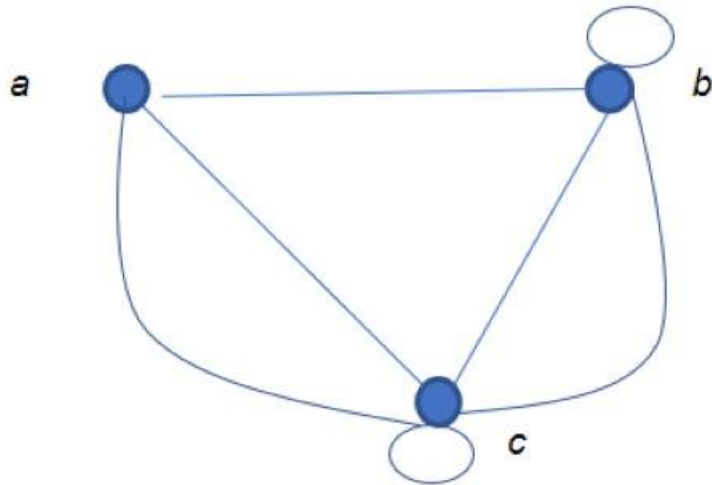
• v

- Match

Trivial Graph

Null Graph

Pseudo Graph



- Solution

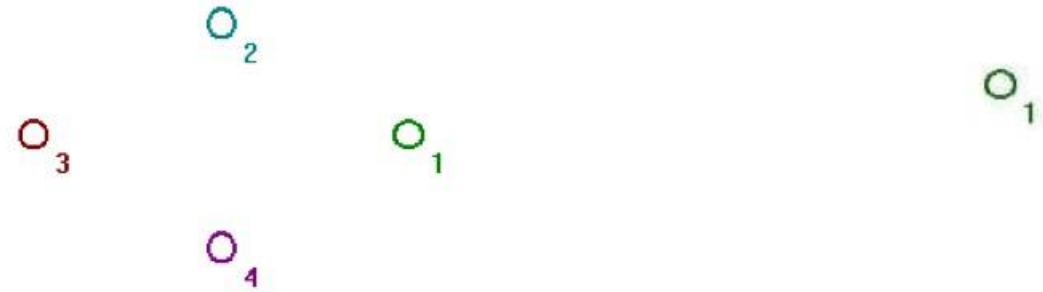
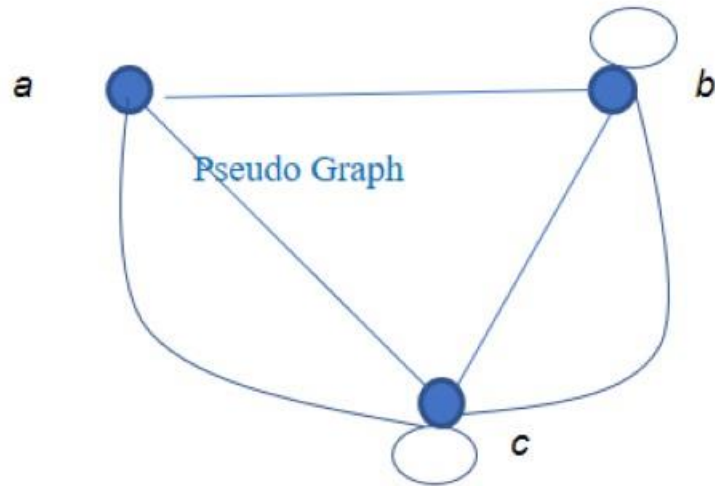
- A. Pseudo Graph
- B. Null Graph
- C. Trivial Graph

- Match

Trivial Graph

Null Graph

Pseudo Graph

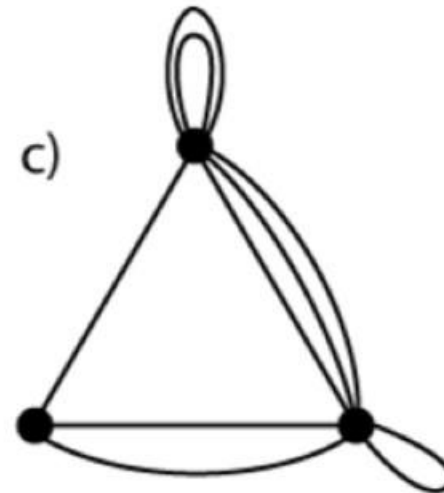
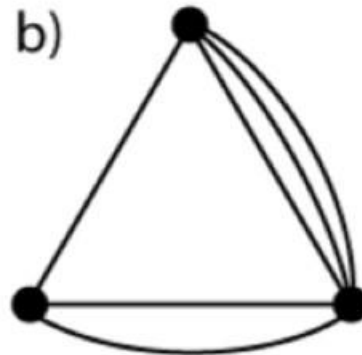
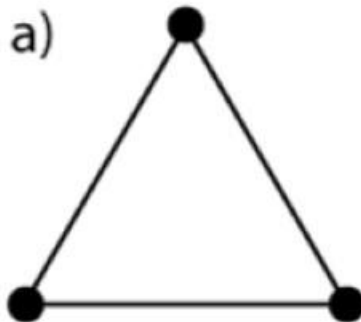


- Match

Simple Graph

multigraph

Pseudo Graph

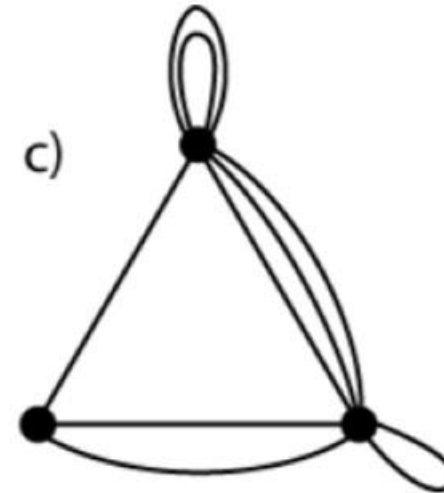
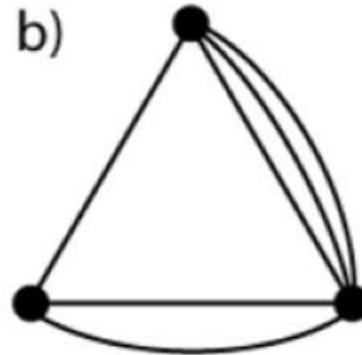
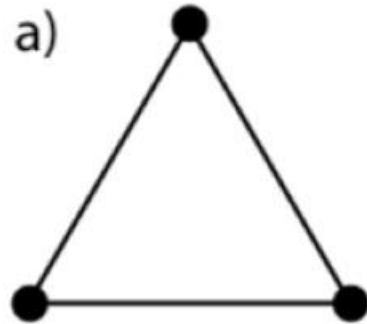


- Match

Simple Graph

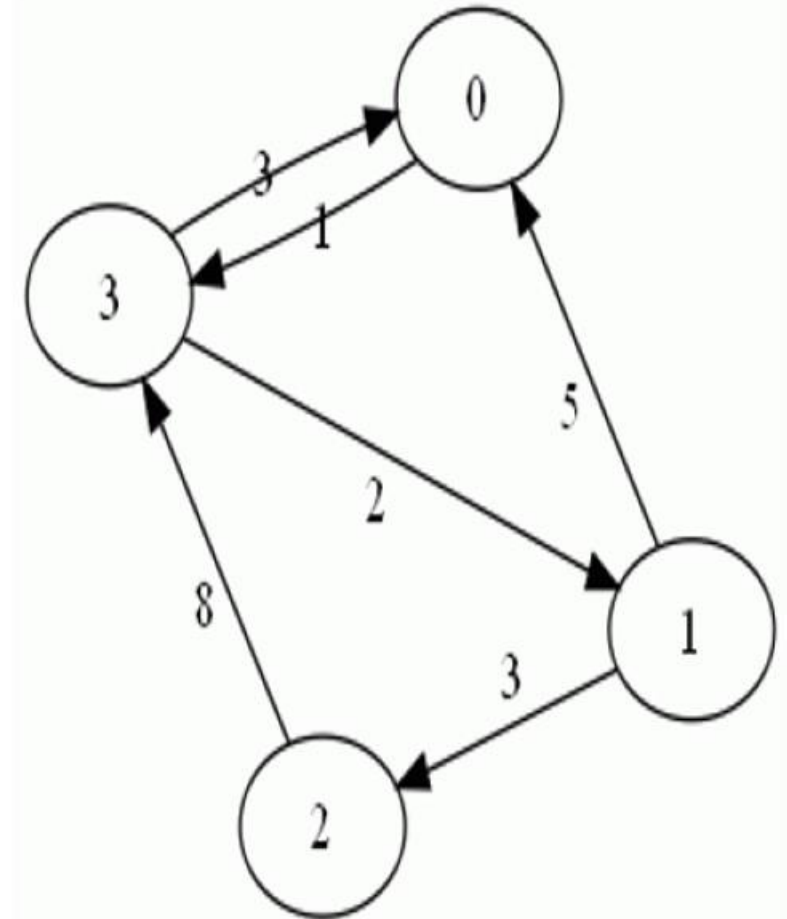
multigraph

Pseudo Graph

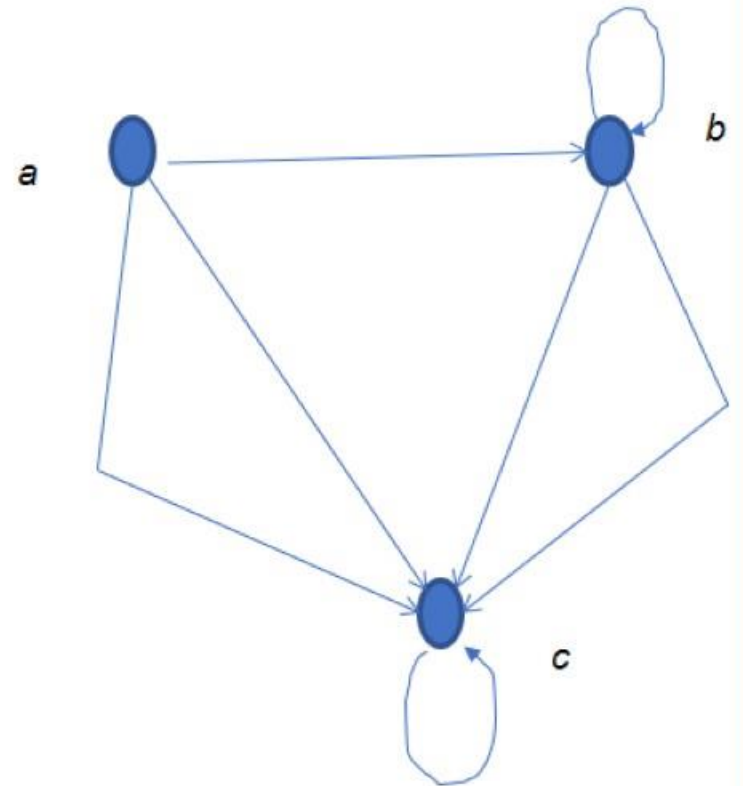


- **Edges** → undirected or directed (or both-Mixed)?
- **Multiple** edges → multigraph?
- **Loops** → Pseudo

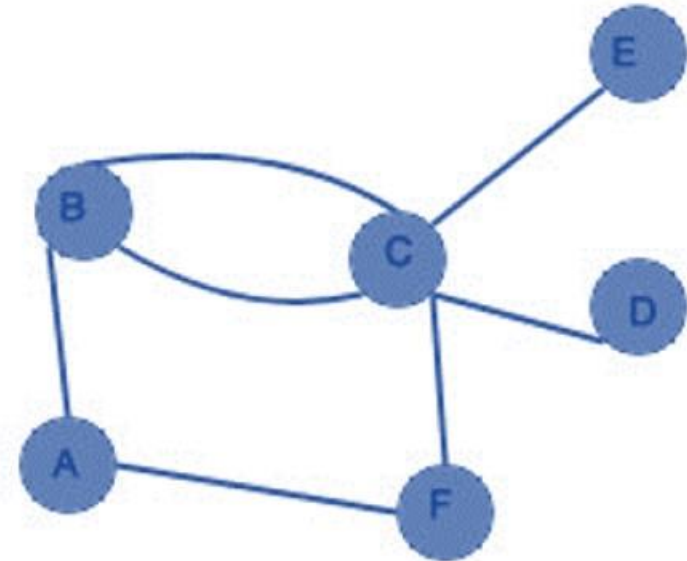
- *Simple directed graph*
 - has no loops and no multiple edges.



- *Pseudo graph*



- **Multiplicity** : the **number of edges** between two vertices.
- The multiplicity of the edge $\{B, C\}$?
- The multiplicity of the edge $\{B, C\}=2$

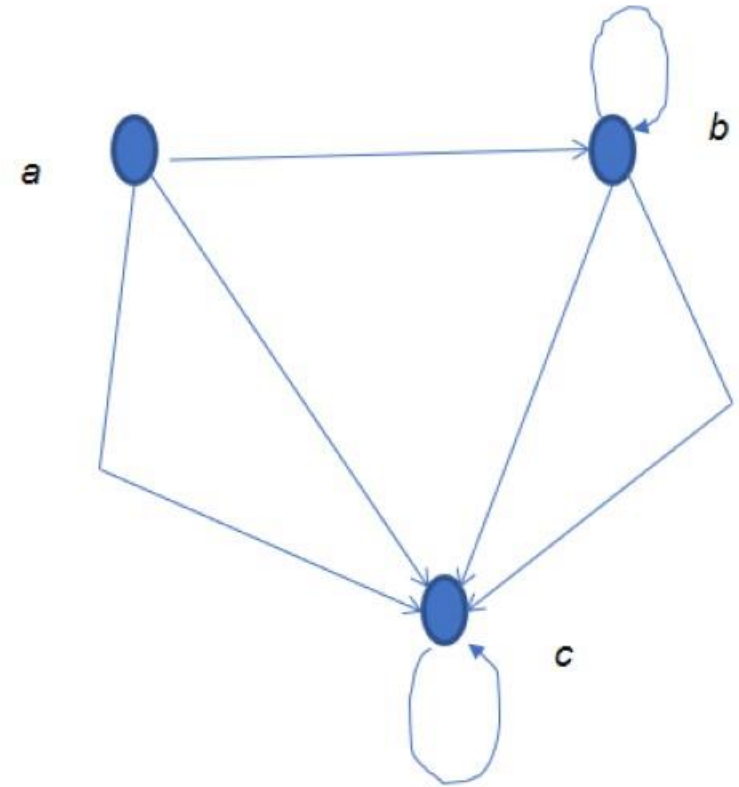


- **Find**
 - **Multiplicity of (a,b)**
 - **Multiplicity of (b,c)**

Multiplicity

- Since there can be multiple edges between the same pair of vertices, the **multiplicity** of edge tells the number of edges between the same pair of vertices.
- The multiplicity of the edge {B, C} is 2

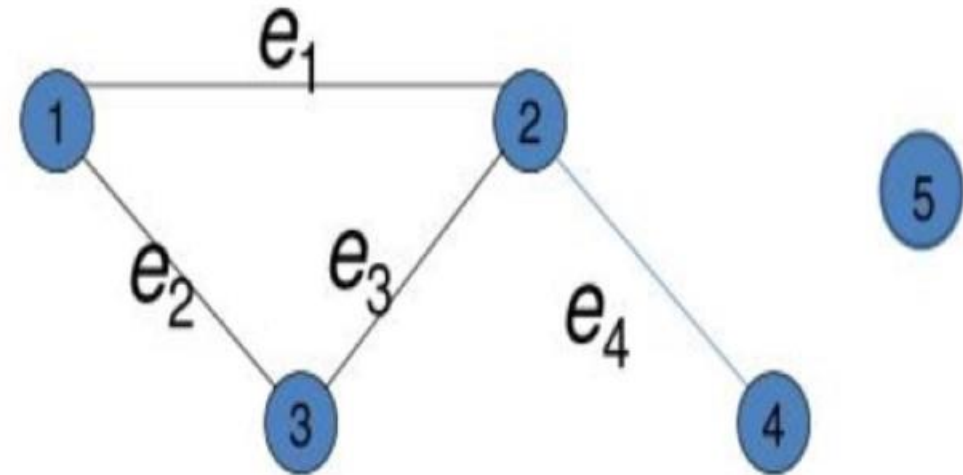
- **Solution**
 - **Multiplicity of (a,b) is 1**
 - **Multiplicity of (b,c) is 2**



Adjacent (or *Neighbors*) in G if there is an edge e between u and v

***Adjacent* (or *Neighbors*)** in G if there is an edge e between u and v

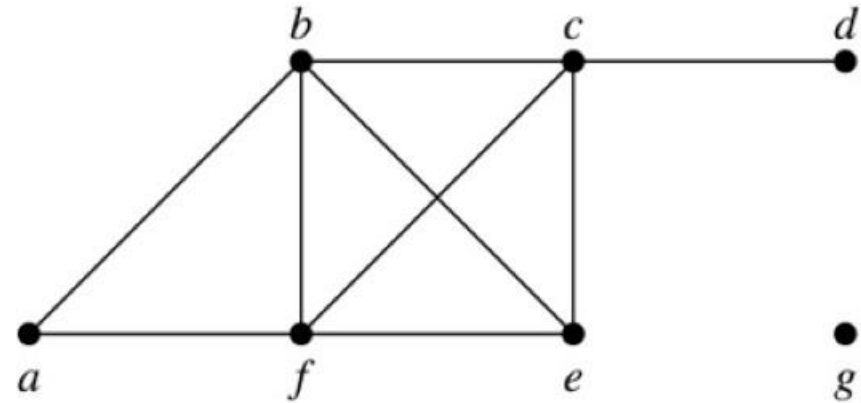
- 1 is adjacent to 2,3
- 2 is adjacent to 1, 3, and 4
- 3 is adjacent to 1 and 2
- 4 is adjacent to 2
- 5 is not adjacent to any vertex



The *degree of a vertex in a undirected graph* is the number of edges incident with it,

The degree of the vertex v is denoted by $\deg(v)$.

Example: What are the degrees of the vertices in the graphs G ?

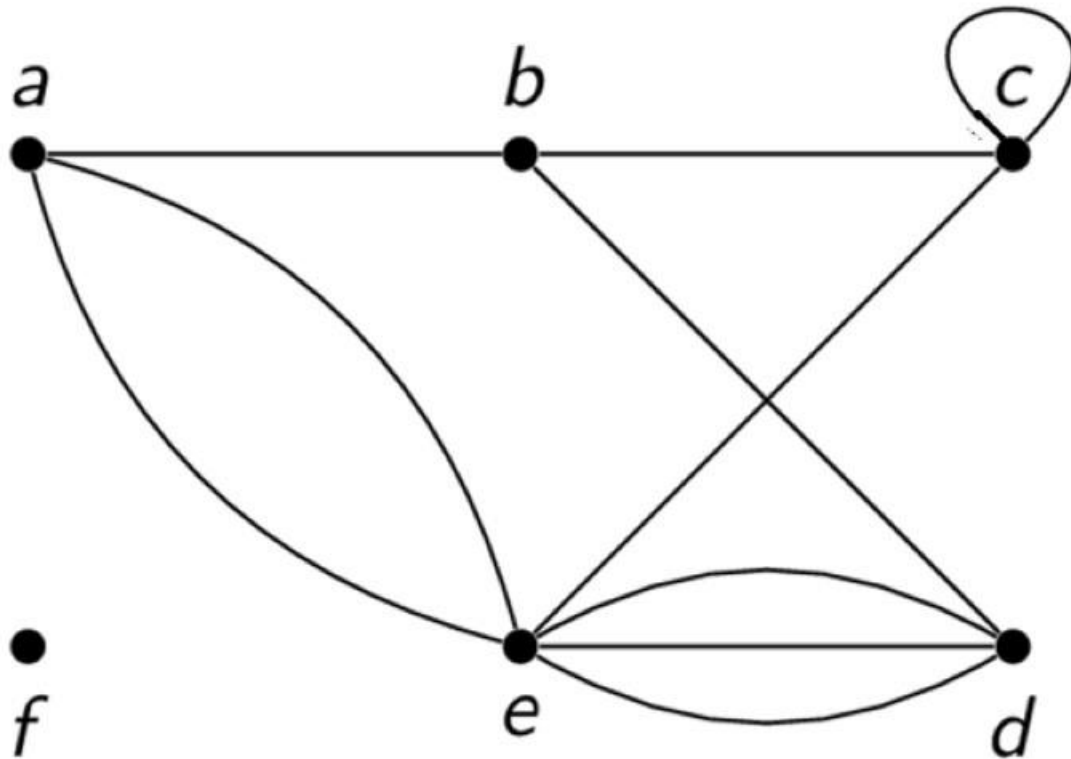


Solution:

G : $\deg(a) = 2, \deg(b) = \deg(c) = \deg(f) = 4, \deg(d) = 1,$
 $\deg(e) = 3, \deg(g) = 0.$

Degree of a vertex

- A **loop** at a vertex contributes **two** to the degree of that vertex.



$$\deg a = 3$$

$$\deg b = 3$$

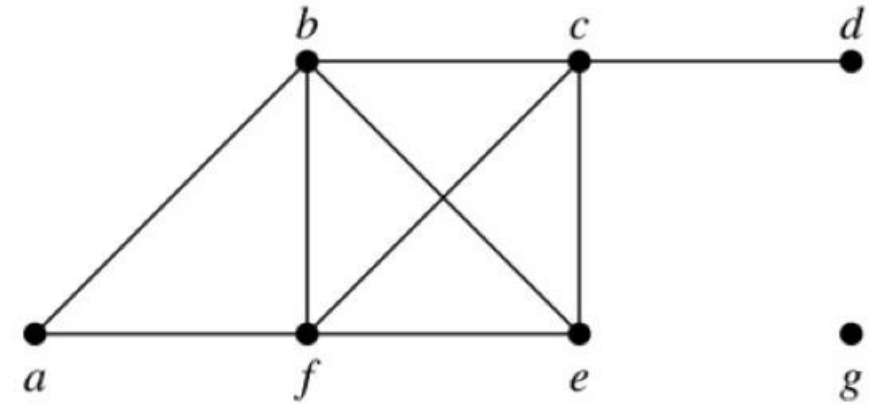
$$\deg c = 4$$

$$\deg d = 4$$

$$\deg e = 6$$

$$\deg f = 0$$

What are neighborhoods of the vertices in the graphs G ?



Solution:

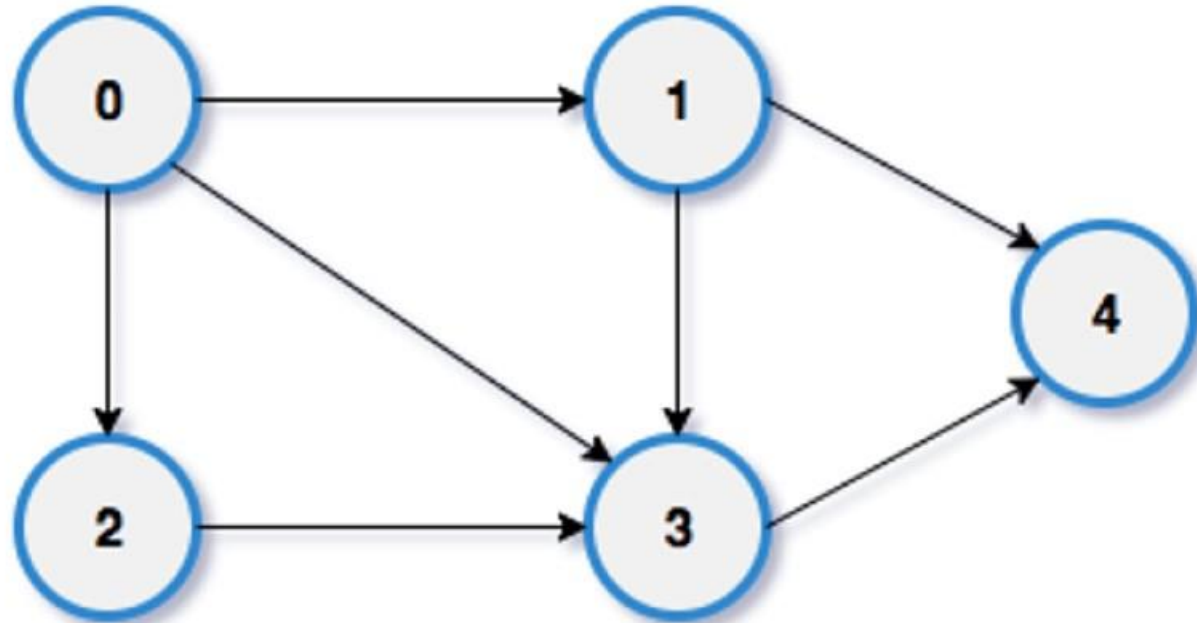
G :

$$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, N(c) = \{b, d, e, f\}, N(d) = \{c\}, \\ N(e) = \{b, c, f\}, N(f) = \{a, b, c, e\}, N(g) = \emptyset.$$

- The *in-degree* of a vertex v , denoted $\deg^-(v)$, is the number of edges which terminate at v .
- The *out-degree* of v , denoted $\deg^+(v)$, is the number of edges with v as their initial vertex.
- **Loop** at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

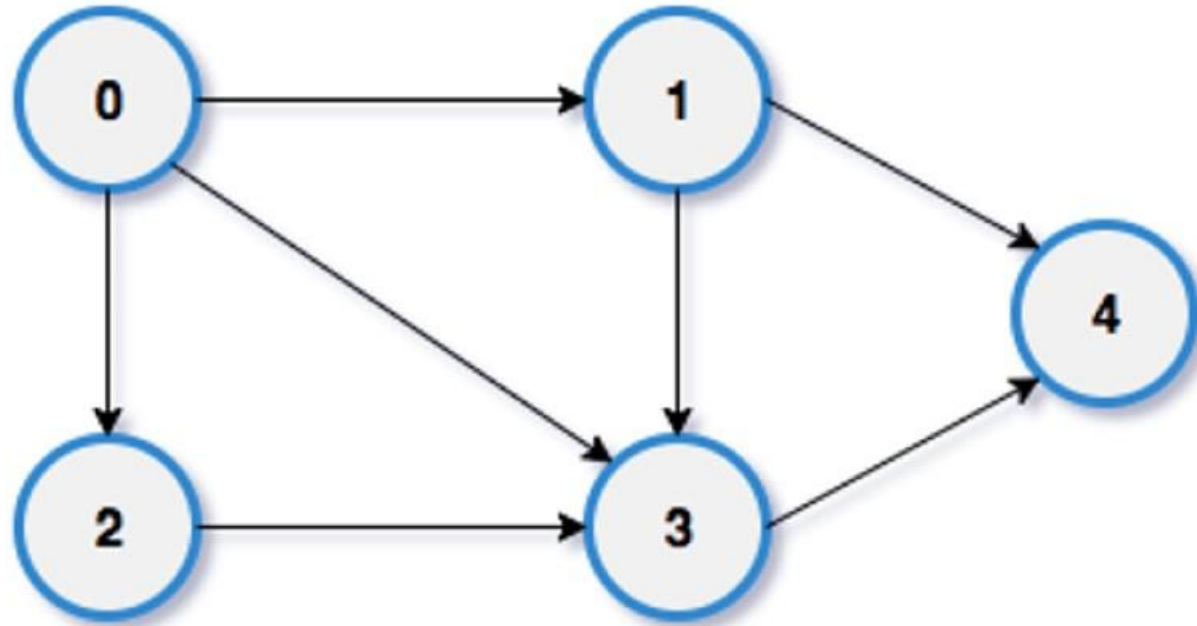
In-degree:

- $\deg^-(0) = 0$
- $\deg^-(1) = 1$
- $\deg^-(2) = 1$
- $\deg^-(3) = 3$
- $\deg^-(4) = 2$

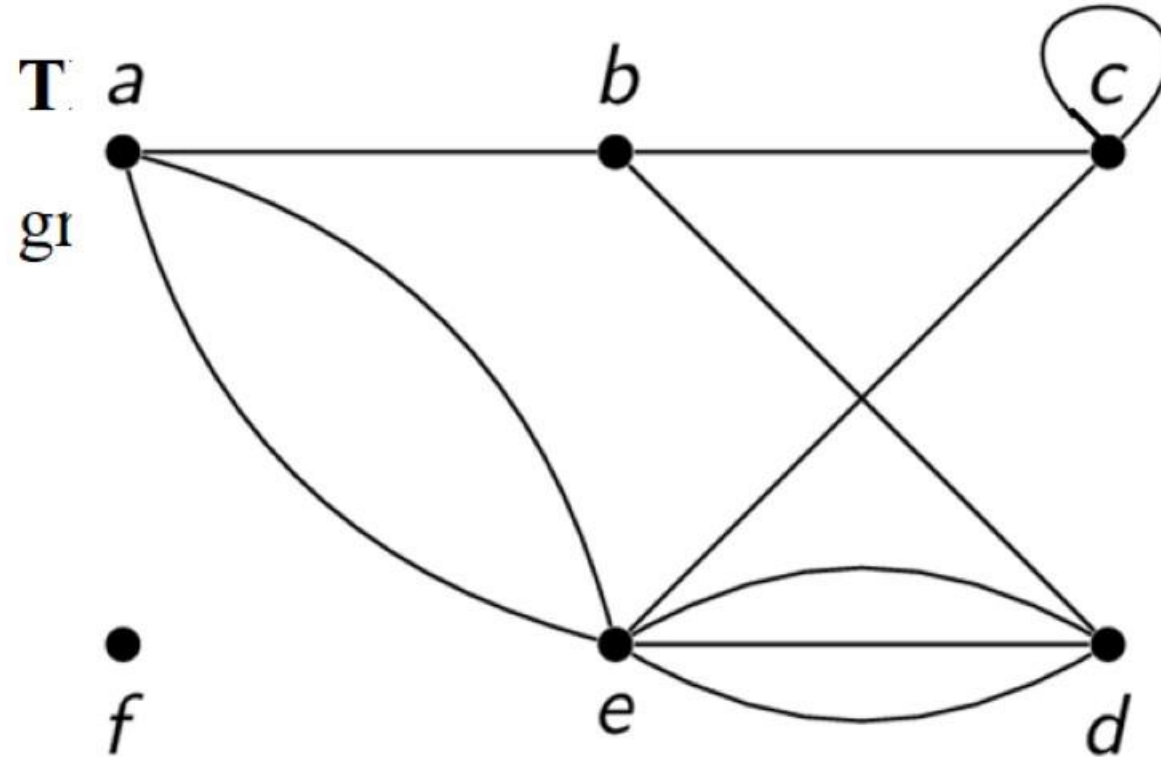


out-degree:

- $\deg^+(0) = 3$
- $\deg^+(1) = 2$
- $\deg^+(2) = 1$
- $\deg^+(3) = 1$
- $\deg^+(4) = 0$



Degrees of Vertices



If $G = (V, E)$ is an undirected graph

$$\deg a = 3$$

$$\deg b = 3$$

$$\deg c = 4$$

$$\deg d = 4$$

$$\deg e = 6$$

$$\deg f = 0$$

$\sum_{v \in V} \deg(v)$

Sum of degrees

How many edges are there in a graph with 10 vertices of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, the handshaking theorem tells us that $2m = 60$.

So the number of edges $m = 30$.

A simple graph G has 24 edges and degree of each vertex is 4. Find the number of vertices.

Solution:

$$2 * \text{number of edges} = \text{sum of degrees}$$

$$2 * 24 = \text{sum of degrees}$$

$$48 = \text{number of vertex} * 4$$

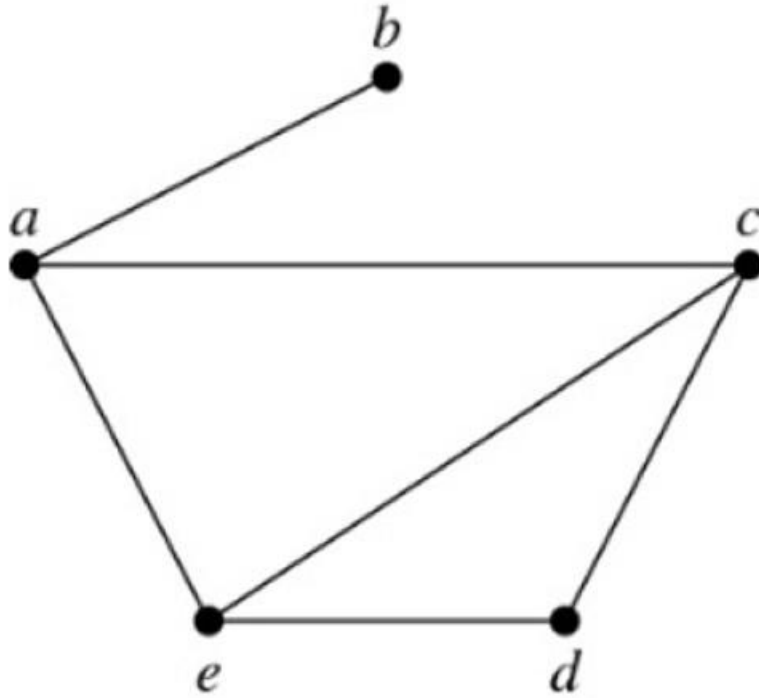
$$\text{number of vertex} = 12$$



Graph Representation

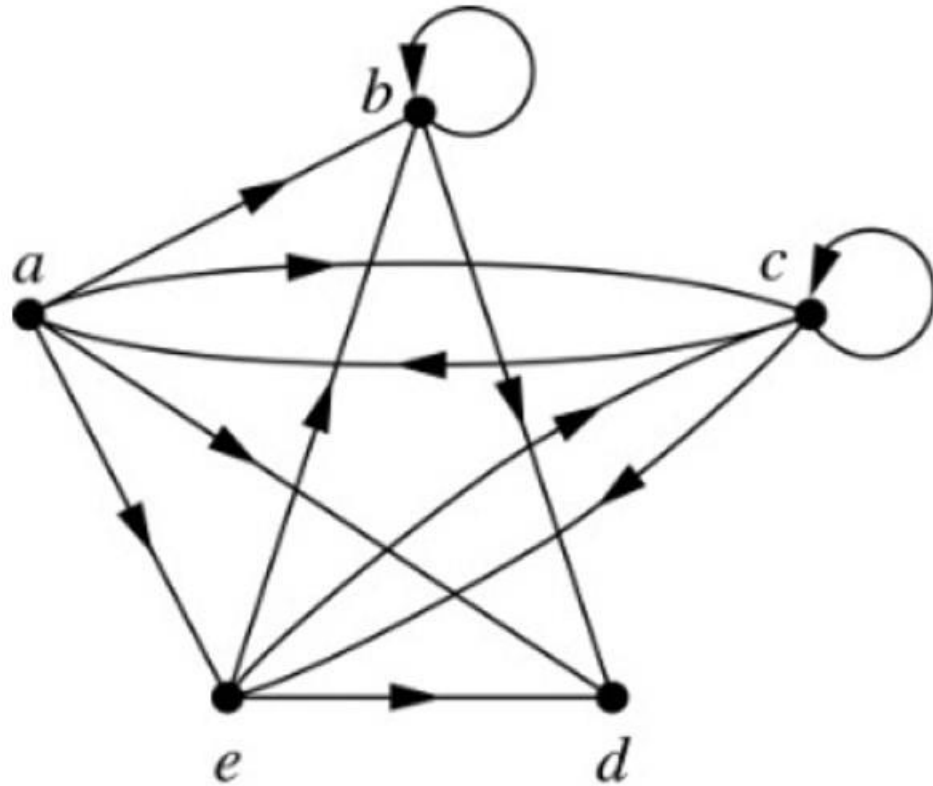
Definition: An *adjacency list* can be used to represent a graph with **no multiple edges** by specifying the vertices that are adjacent to each vertex of the graph.

Representing Graphs: Adjacency Lists



<i>Vertex</i>	<i>Adjacent Vertices</i>
<i>a</i>	<i>b, c, e</i>
<i>b</i>	<i>a</i>
<i>c</i>	<i>a, d, e</i>
<i>d</i>	<i>c, e</i>
<i>e</i>	<i>a, c, d</i>

Representing Graphs: Adjacency Lists



<i>Initial Vertex</i>	<i>Terminal Vertices</i>
<i>a</i>	<i>b, c, d, e</i>
<i>b</i>	<i>b, d</i>
<i>c</i>	<i>a, c, e</i>
<i>d</i>	
<i>e</i>	<i>b, c, d</i>

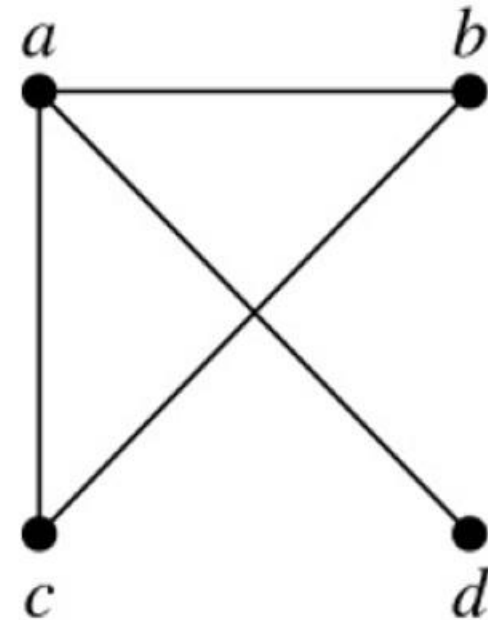
Definition: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. The *adjacency matrix* $\mathbf{A}_G$ of G the $n \times n$ zero-one matrix with:

- 1 as its (i, j) th entry when v_i and v_j are adjacent,
- 0 as its (i, j) th entry when they are not adjacent.
- In other words, if the graphs adjacency matrix is

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

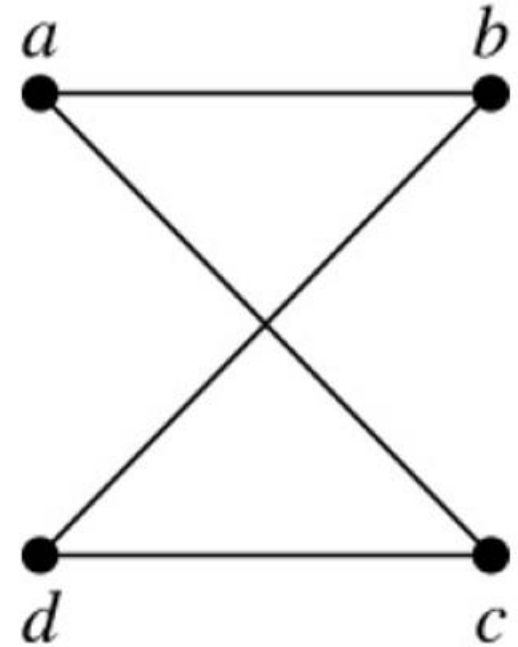
Represent the Following Graph using Adjacency Matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

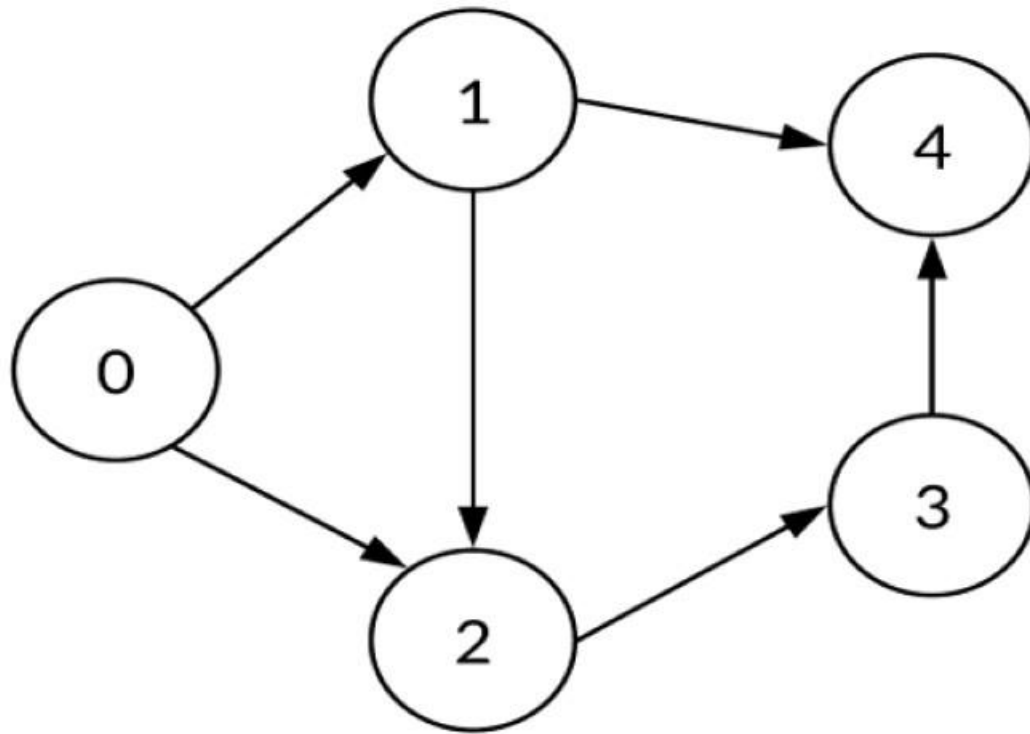


- Represent the Following Graph using Adjacency Matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Represent the Following Graph using Adjacency Matrix

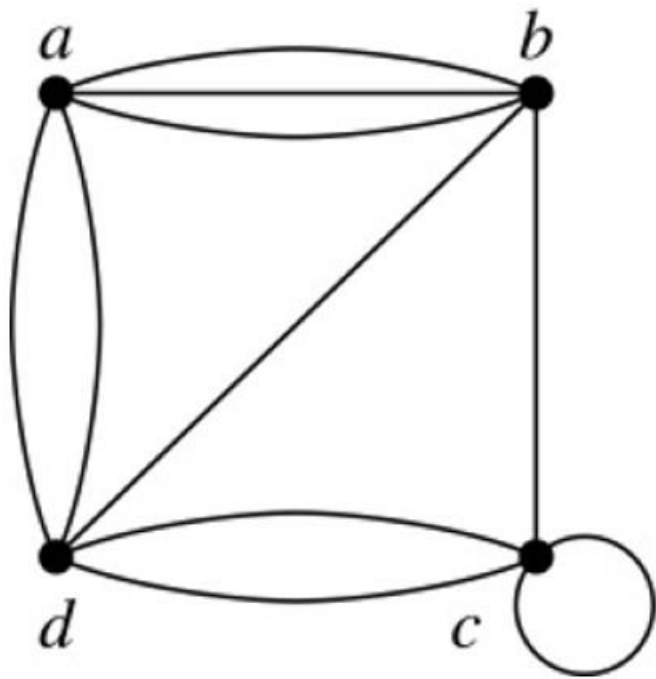


Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

- Adjacency matrices can also be used to represent graphs with loops and multiple edges.
- When **multiple** edges connect the same pair of vertices or if multiple **loops** are present at the same vertex , the (i, j) th entry equals the number of edges connecting the pair of vertices.

Example: We give the adjacency matrix of the pseudograph shown here using the ordering of vertices a, b, c, d .



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

- Graph Type

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

- Graph Type

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Multiedge
Loops

undirected

Adjacency Matrix

	0	1	2	3	4
0	0	1	1	0	0
1	0	0	1	0	1
2	0	0	0	1	0
3	0	0	0	0	1
4	0	0	0	0	0

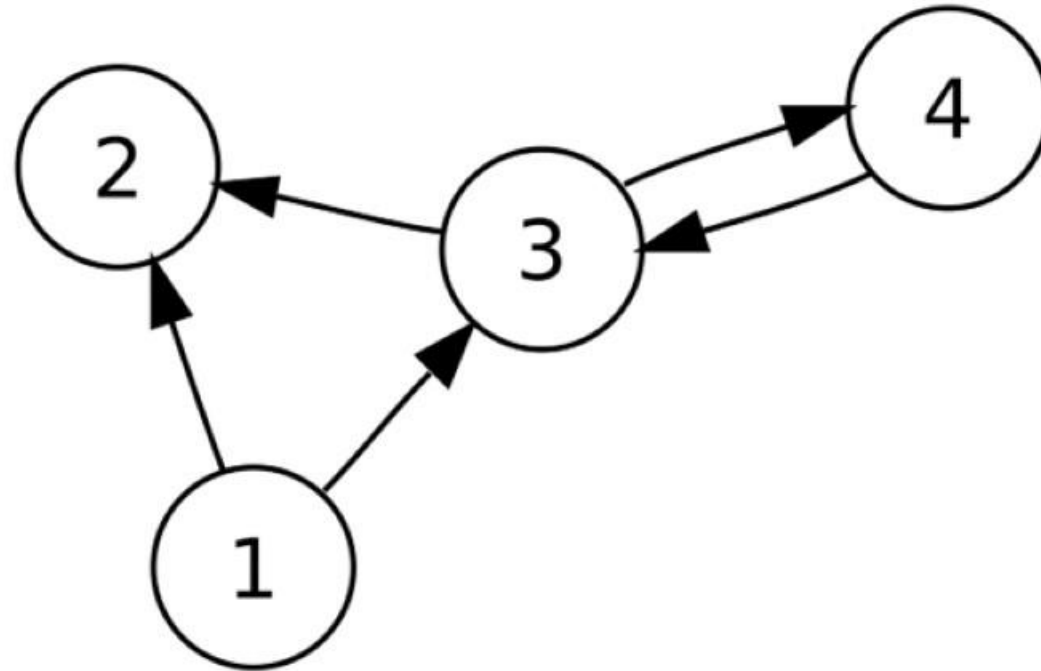
Directed



Recap

- True or False

- $(1,2) \in G$
- $(3,2) \in G$
- $(2,2) \in G$
- $(2,1) \in G$

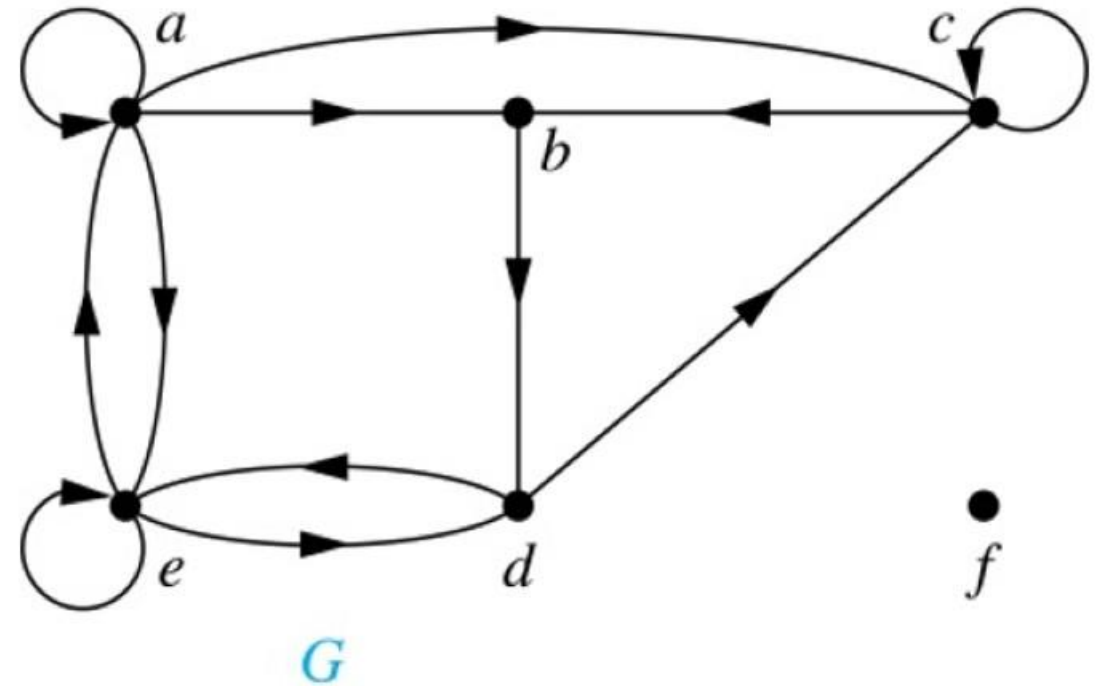


Solution

True, True, False, False

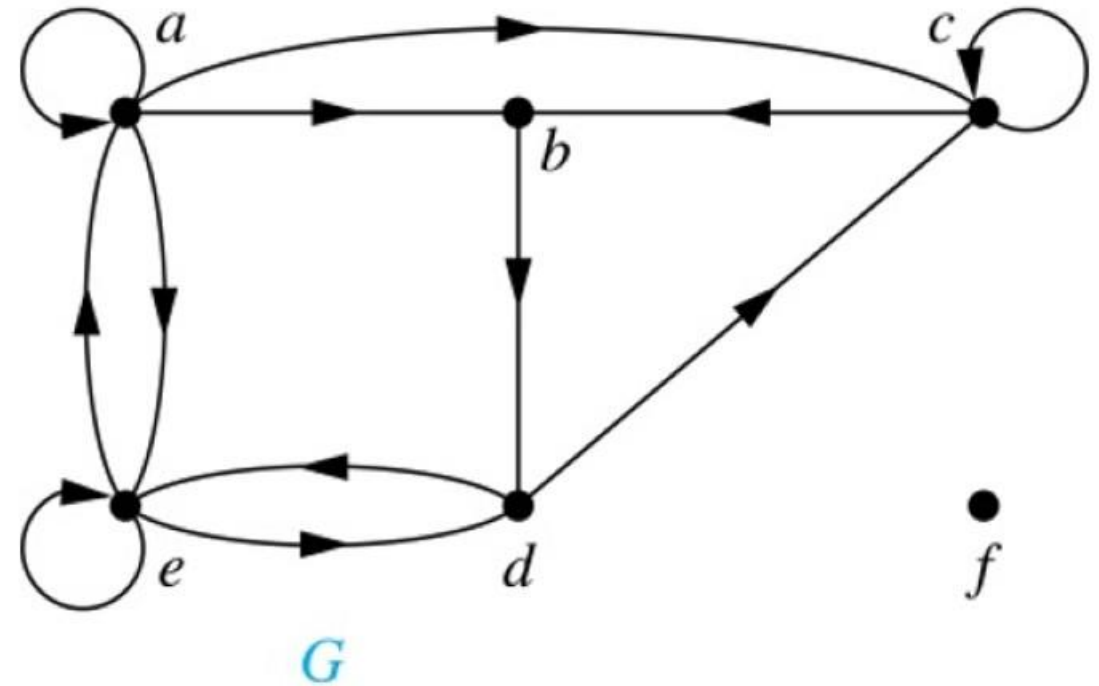
In the graph *Find all indegree and out degree for all nodes*

$\deg^-(a) = 2$	$\deg^+(a) = 4$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2,$	$\deg^+(d) = 2$
$\deg^-(e) = 3$	$\deg^+(e) = 3$
$\deg^-(f) = 0.$	$\deg^+(f) = 0.$

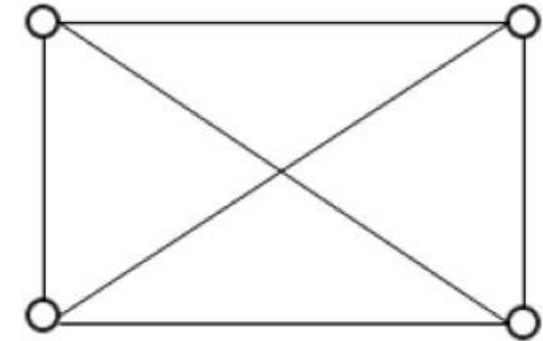
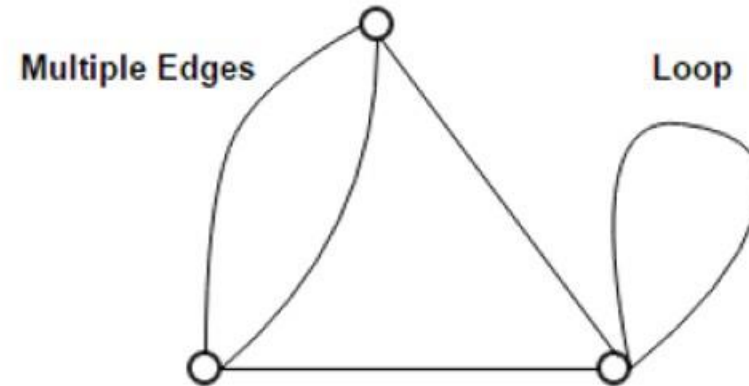


In the graph *Find all indegree and out degree for all nodes*

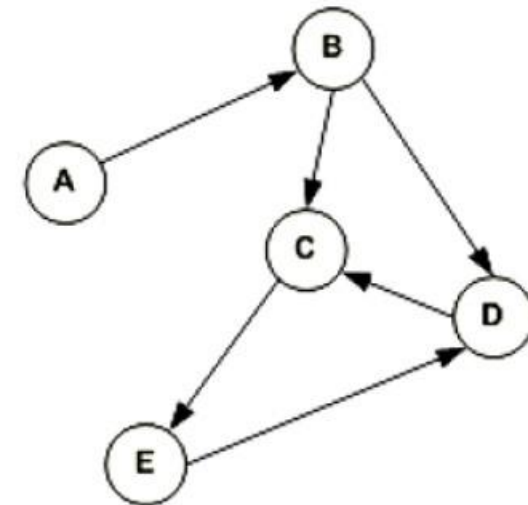
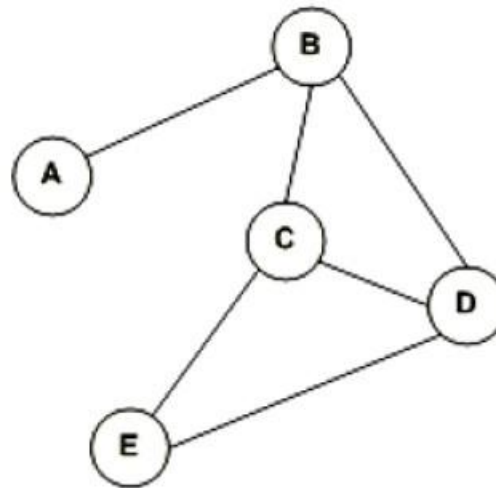
$\deg^-(a) = 2$	$\deg^+(a) = 4$
$\deg^-(b) = 2$	$\deg^+(b) = 1$
$\deg^-(c) = 3$	$\deg^+(c) = 2$
$\deg^-(d) = 2,$	$\deg^+(d) = 2$
$\deg^-(e) = 3$	$\deg^+(e) = 3$
$\deg^-(f) = 0.$	$\deg^+(f) = 0.$



Simple Graph or not?

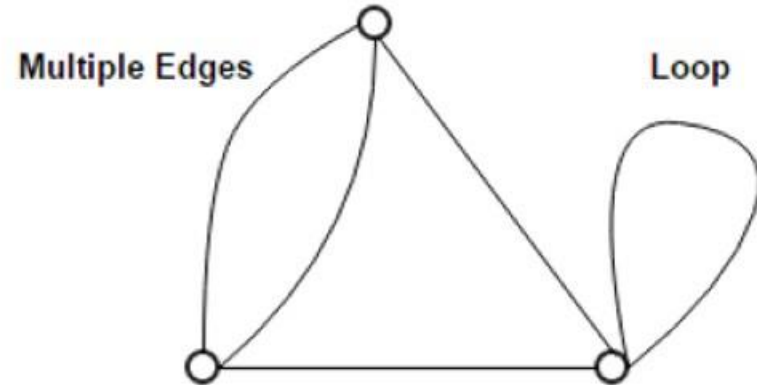


Directed or undirected ?

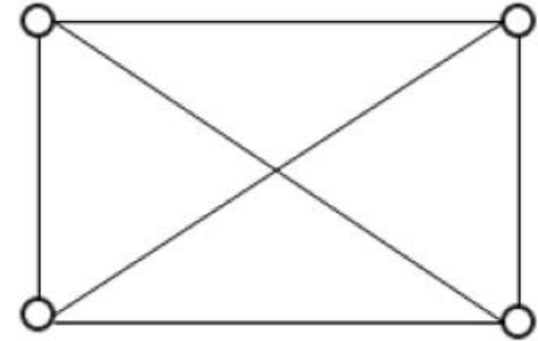


Simple Graph or not?

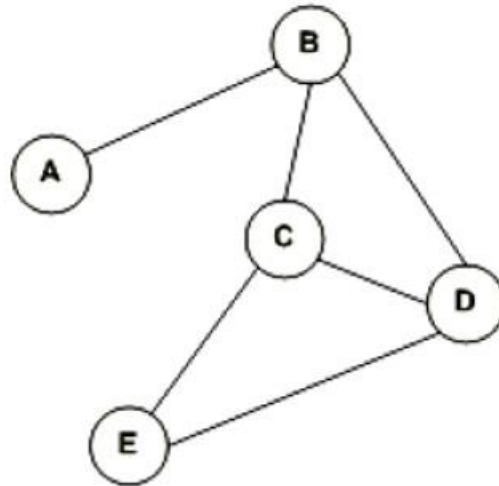
not simple graph



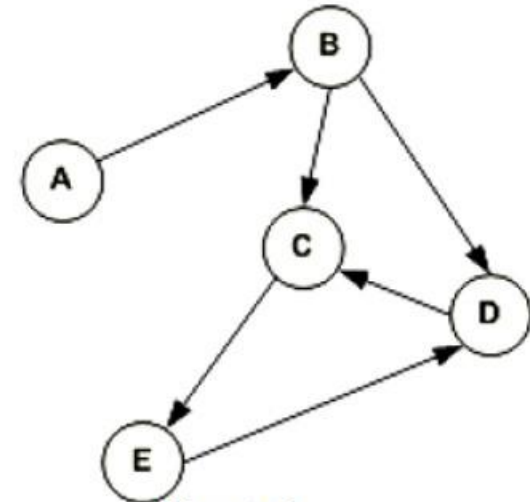
Simple Graph



Directed or undirected ?

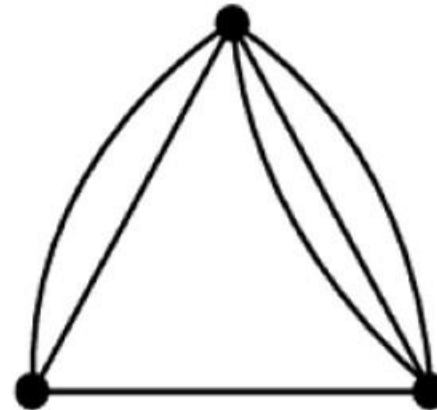
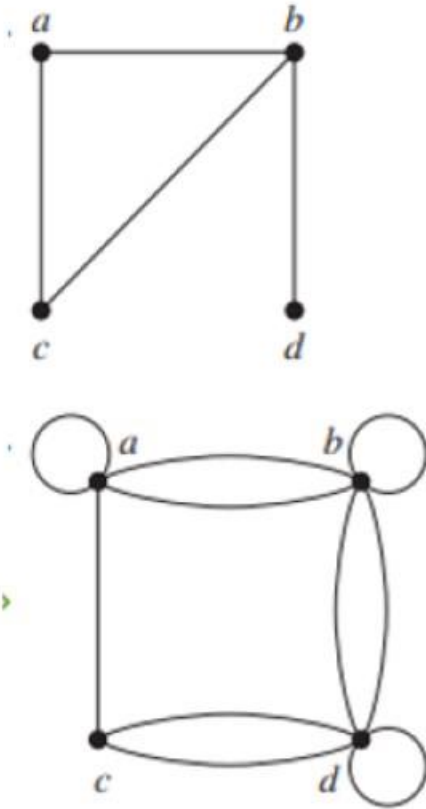


Undirected



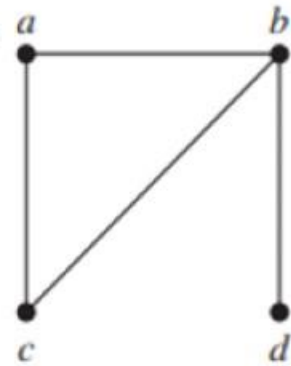
Directed

- State type of graph (Simple-multigraph-pseudograph)

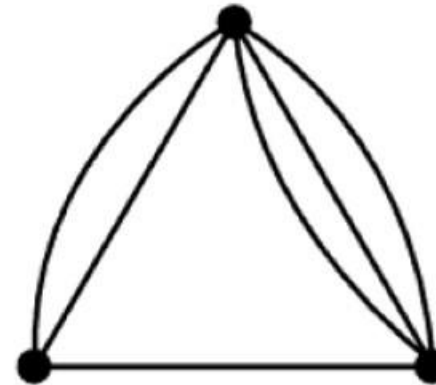
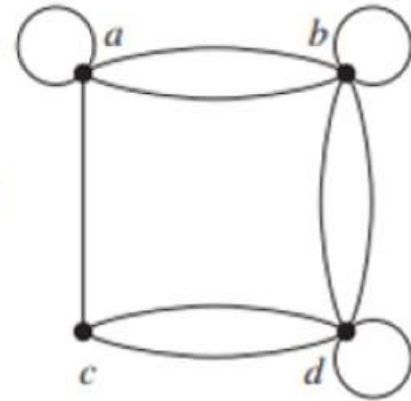


State the type of graph in the following figures

Simple
graph

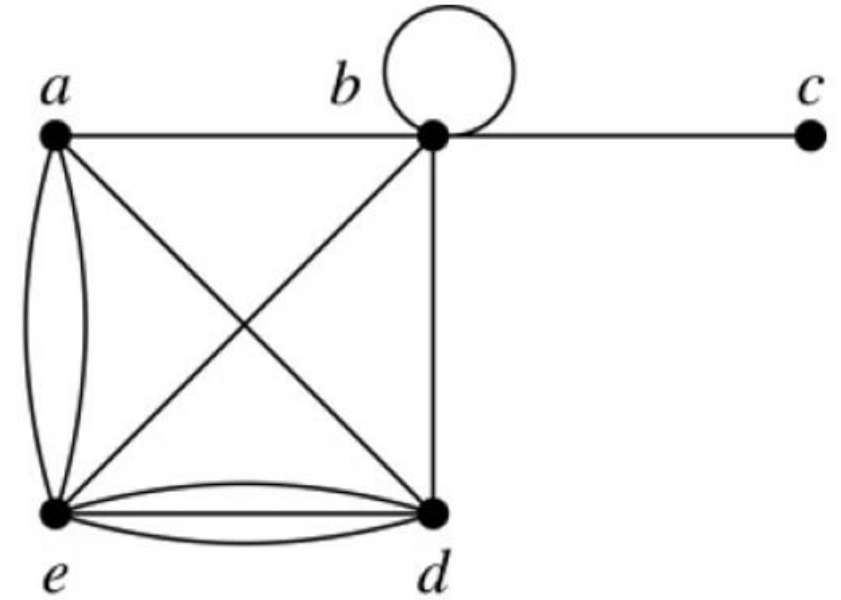


Pseudo
graph



multigraph

Example: What are the degrees and neighborhoods of the vertices in the graphs H ?



Solution:

$$\deg(a) = 4, \deg(b) = \deg(e) = 6, \deg(c) = 1, \\ \deg(d) = 5.$$

$$N(a) = \{b, d, e\}, N(b) = \{a, b, c, d, e\}, N(c) = \{b\},$$

$$N(d) = \{a, b, e\}, N(e) = \{a, b, d\}.$$

H

How many edges are there in a graph with 10 vertices of degree Five?

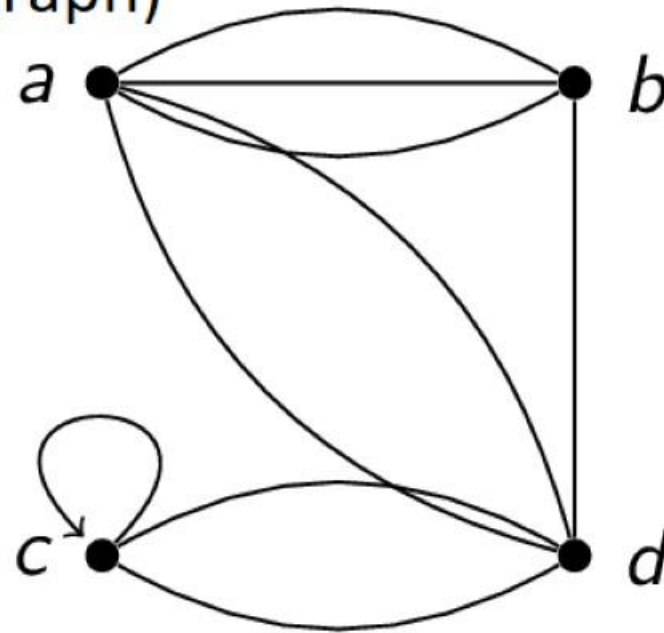
Solution: Because the sum of the degrees of the vertices is $5 \cdot 10 = 50$, the handshaking theorem tells us that $2m = 50$.

So the number of edges $m = 25$.

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{pmatrix} a & b & c & d \\ 0 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

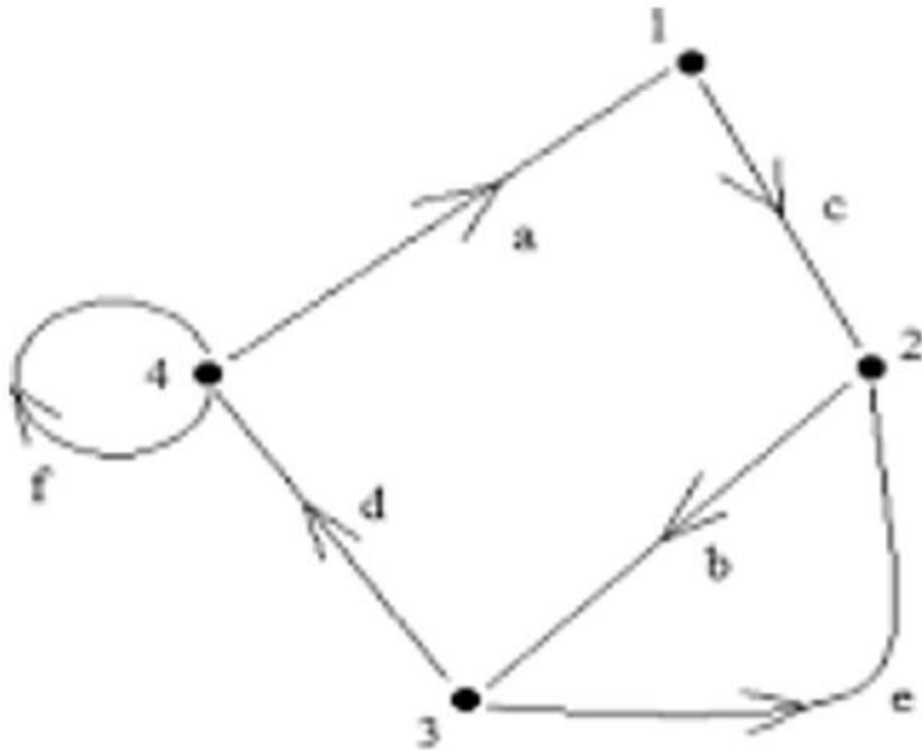


$$\begin{matrix} & a & b & c & d \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \end{matrix}$$

Multigraph loop and undirected

- According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

$$M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$



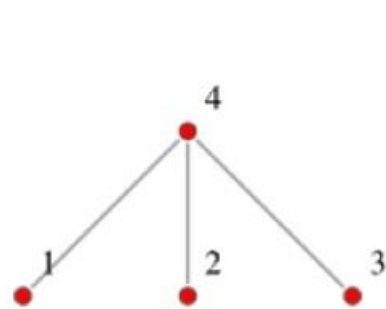
$$M = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

According to Adjacency matrix, the graph is (directed-undirected-has loops-multigraph)

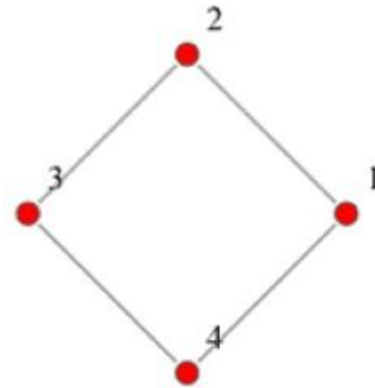
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

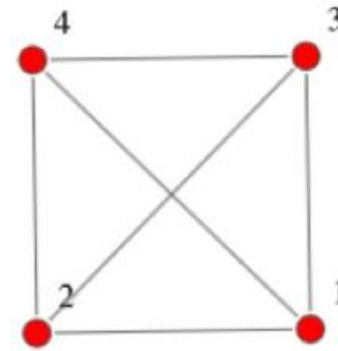
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Simple graph

Complete

1. There are no edges between its vertices.
2. Edge set is empty
3. The graph which has only one vertex
4. We use ordered pair to represent ----- graph
5. We use sets to represent edge -----graph
6. A loop at a vertex contributes _____ to the degree of that vertex.
7. An *adjacency list* can be used to represent a graph with multiple edges?

1. Null graph
2. Null graph
3. Trivial graph
4. Directed
5. Undirected
6. Two
7. False

- Graph
- Types of graph
- Degrees
- Neighborhood
- Adjacency List
- Adjacency Matrix



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