

Financial mathematics

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Lecture 09



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EVERYONE'S SMART UNIVERSITY

MTH 112
Business Mathematics

Chapter 05

At the end of this section you should be familiar with:

- Using the net present value as an investment appraisal technique
- Using the internal rate of return as an investment appraisal technique
- Comparing the two techniques.
- Annuity

Net present value (NPV)

The net present value is the present value of several future sums discounted back to the present.

The net present value technique uses present values to appraise the profitability of investments undertaken by firms.

Net present value (NPV)

The problem confronting the project manager is whether the investment is worth while. The decision is made even more difficult since the returns associated with the project will be received in the future while the costs usually occur at present.

Net present value (NPV)

Therefore, to make an informed decision, the project manager needs to compare costs and returns, all brought back to present values. These present returns are then compared to present costs to estimate a net present value calculation on the project where

$$\text{NPV} = \text{present value of cash inflows} - \text{the present value of cash outflows} \quad (5.18)$$

A decision rule is

$\text{NPV} > 0$ Invest in the project

$\text{NPV} < 0$ Don't invest in the project

Example:

Calculate the net present value of the investment project as outlined in Table 5.3 assuming 8% discount rate compounded annually.

Note:

When future values are brought back to present values at a given rate of interest, the interest rate is referred to as the discount rate.

Table 5.3 Cash flows of an investment project

Year	Cash flows (£000)
0	-400
1	120
2	130
3	140
4	150

Solution:

The present value of the cash outflow is the initial outlay of £400 000. The present value of cash inflows is calculated by summing the present value of each cash inflow as follows:

Present value of return in year 1:
$$P_0 = \frac{120\,000}{(1 + 0.08)^1} = 111\,111$$

Present value of return in year 2 :
$$P_0 = \frac{130\,000}{(1 + 0.08)^2} = 111\,454$$

Solution:

Present value of return in year 3 :

$$P_0 = \frac{140\,000}{(1 + 0.08)^3} = 111\,137$$

Present value of return in year 4 :

$$P_0 = \frac{150\,000}{(1 + 0.08)^4} = 110\,254$$

Solution:

Total present value over the four years: **443 956**

The net present value is then

$$\text{NPV} = 443\,956 - 400\,000 = 43\,956$$

As the NPV is positive, the firm will invest in this project. Thus, the investment is cost effective at 8% discount rate compounded annually.

The size of the NPV will become smaller when

- (i) The initial cost outlay increases
- (ii) The discount rate at which NPV is computed increases

Internal rate of return (IRR):

In the previous section, the profitability of a project was assessed on the basis of the **NPV** calculated at a given rate of interest of **8%**. The profitability changes as the interest rate changes. If the interest rate was **12.6555%**, the NPV would be zero, while further increases in the interest rate, such as **$r = 15\%$** , would result in negative NPVs. Table 5.4 shows the calculation of the NPV for these three interest rates.

Table 5.4 Calculation of *NPVs* for various interest rates

<i>t</i> years	Net cash flow	Dis. factor <i>i</i> = 8% $\frac{1}{(1 + 0.08)^t}$	Present value (8%)	Dis. factor 12.6555% $\frac{1}{(1 + 0.126555)^t}$	Present value (12.6555%)	Dis. factor <i>i</i> = 15% $\frac{1}{(1 + 0.15)^t}$	Present value (15%)
0	-400	1	-400	1	-400	1	-400
1	120	0.9259	111	0.8877	107	0.8696	104
2	130	0.8573	111	0.7879	102	0.7561	98
3	140	0.7938	111	0.6994	98	0.6575	92
4	150	0.7350	110	0.6209	93	0.5718	86
			43		0		-20

Internal rate of return (IRR):

The IRR is the interest rate for which the NPV is zero. A project is viable if the prevailing interest rate is less than the IRR, but not profitable if the prevailing interest rate is greater than the IRR.

The decision rule is :

Accept the project if the IRR is greater than the market rate of interest.

To calculate the IRR for a given project:

To calculate the IRR for a given project:

- (A) graphically,
- (B) by calculation

(A) The graphical method: The NPV for several rates of interest is plotted against the interest rates, such that the NPVs range from positive to negative. Then, according to the definition given above, the IRR is the interest rate at which the $NPV = 0$. This is the point at which the curve crosses the horizontal axis.

To calculate the IRR for a given project:

To calculate the IRR for a given project:

- (A) graphically,
- (B) by calculation

(B) By calculation: As a follow-on from the graphical method described above, the point may be calculated at which the curve of NPV plotted against i crosses the horizontal axis.

To calculate the IRR for a given project:

To calculate the IRR for a given project:

- (A) graphically,
- (B) by calculation

(B) By calculation: If two points on the curve are known, one above the axis (i_1, NPV_1) where the $NPV_1 > 0$ and the other below the axis (i_2, NPV_2) where the $NPV_2 < 0$, the IRR, which is the point of intersection with the horizontal axis, is given by the formula.

$$IRR = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} \quad (5.19)$$

Example: A project involves an initial outlay of £550 000. The expected cash flow at the end of the next five years is given as

Year	1	2	3	4	5
Cash flow	<u>108 300</u>	<u>130 500</u>	<u>170 900</u>	200 000	160 000

Determine the IRR

- (A) Graphically by plotting the NPV against i for $i = 0.05; 0.07; 0.09; 0.11; 0.13; 0.15; 0.17$
- (b) By calculation. Show that the value of the IRR is slightly different when calculated from different pairs of points.

(A) Graphically: Excel is ideal at this point, since the calculation of NPVs requires the repeated use of formulae. For part (A), several NPVs are required to plot the curve of NPV against i . The formula which calculates the present value of the cash flow at discount rate i , expressed as a decimal, is entered into the appropriate cell as follows:

$$= (\text{cash flow}) * (1/(1+i)^t)$$

Since the values of t are always in column A and cash flow in column B (Table 5.5), the addresses of these columns may be entered in the following formula at cell C2:

$$= (\$B2) * (1/(1+i)^{\$A2})$$

$$P_0 = \frac{P_t}{(1+i)^t}$$

- Enter this formula in cell C2, with $(1 + i) = 1.05$. Then copy the formula down column C to calculate the present values for years 1 to 5. Point to cell C8. Then use the Σ button to sum the contents of cells C2 to C7 to find the NPV.
- To calculate the NPV for each of the other discount rates, copy the formula in cell C2 across row 2. The \$ sign in front of the A and B means that these references do not change.

However, since the discount rate is different for each column, this part of the formula must be changed. For example, in the 11 % column, replace $(1 + i)$ by 1.11.

- Repeat the 'copy down and sum' procedure described above for columns D, E, F, G, H and I in Table 5.5.

The NPV for each discount rate is plotted in Figure 5.1. The IRR is the value of i at which this graph crosses the horizontal axis.

In Figure 5.1, this point is between $i = 11\%$ and 13% , but considerably closer to 11% . IRR $\cong 11.45\%$.

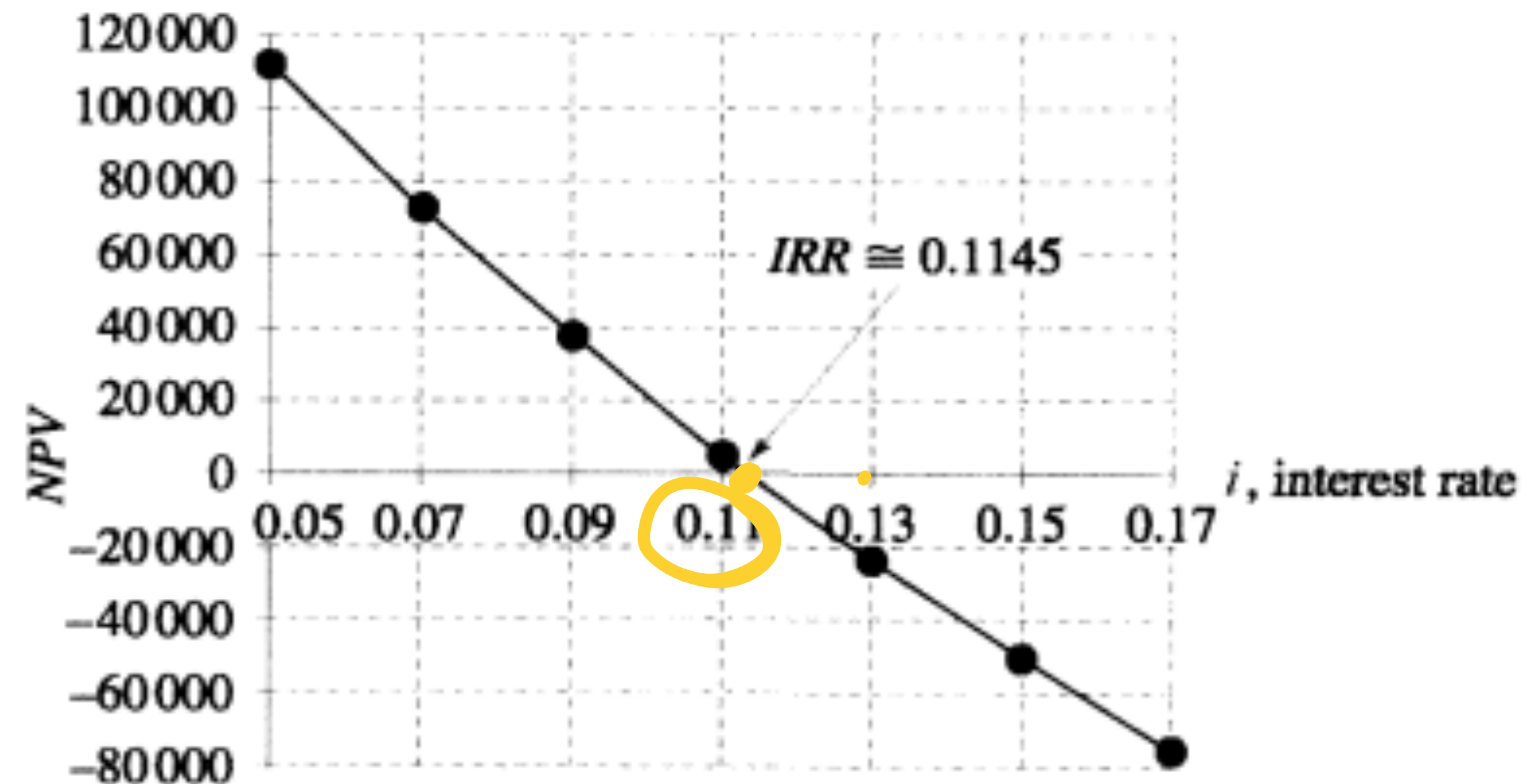
$$P_0 = \frac{P_t}{(1+i)^t}$$

Table 5.5 Excel sheet for calculating *NPV*s at different interest rates

	A	B	C	D	E	F	G	H	I
1	<i>t</i>	cash flow	<i>i</i> = 0.05	<i>i</i> = 0.07	<i>i</i> = 0.09	<i>i</i> = 0.11	<i>i</i> = 0.13	<i>i</i> = 0.15	<i>i</i> = 0.17
2	0	-550000	-550000	-550000	-550000	-550000	-550000	-550000	-550000
3	1	108300	103143	101215	99358	97568	95841	94174	92564
4	2	130500	118367	113984	109839	105917	102201	98677	95332
5	3	170900	147630	139505	131966	124961	118442	112370	106705
6	4	200000	164540	152579	141685	131746	122664	114351	106730
7	5	160000	125364	114078	103989	94952	86842	79548	72978
8	<i>NPV</i> =		109045	71361	36837	5143	-24011	-50881	-75691

$$= (\$B2) \times (1 / (1+i)^A2)$$





Note:

For convenience, the points are labelled according to the column headings in Table 5.5

By calculation:

In Table 5.5 there are several positive and negative NPVs.

Therefore, the IRR given by formula (5.19) is calculated from any such pair.

Summarizing the points already calculated.

Points	C	D	E	F	G	H	I
i	0.05	0.07	0.09	0.11	0.13	0.15	0.17
NPV	109045	71 361	36 837	5143	-24011	-50 881	-75 691

(i) From points C and G

$$\text{IRR} = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} = \frac{(0.05)(-24011) - (0.13)(109045)}{-24011 - 109045} = \frac{-15376.4}{-133056} = 0.11556$$

Therefore, IRR = 11.556%

(ii) From points F and I

$$\text{IRR} = \frac{(i_1 \times NPV_2) - (i_2 \times NPV_1)}{NPV_2 - NPV_1} = \frac{(0.11)(-75691) - (0.17)(5143)}{-75691 - 5143} = \frac{-9200.32}{-80834} = 0.1138175$$

Therefore, **IRR = 11.382%**

The Two results (i) and (ii) demonstrate that slightly different estimates are calculated from different pairs of points.

Comparison of appraisal techniques: NPV, IRR

When comparing the profitability of two or more projects, the most profitable project would be (a) the project with the largest NPV, (b) the project with the largest IRR.

The advantage of the NPV method is that it gives results in cash terms; it is practical as it discounts net cash flows. A disadvantage is that it relies on the choice of one discount rate; it is possible that a change in the discount rate could lead to a change in the choice of project.

Comparison of appraisal techniques: NPV, IRR

The IRR method of appraisal does not depend on any external rates of interest, but a major disadvantage of this method is that it does not differentiate between the scale of projects; for example, one project might involve a cash flow in units of £100000 while another might involve units of £1.

Compound interest for fixed deposits at regular intervals of time

the standard compound interest formula $P_t = P_0(1 + i)^t$ was derived to calculate the value of an investment of P_0 after t years when interest is compounded annually at $i\%$ per annum.

We now consider the situation where, in addition to the initial investment of P_0 , a fixed amount A_0 is deposited at the end of each year for a period of t years. The following table sets out the calculation of the value of such an investment, year by year.

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Compound interest for fixed deposits at regular intervals of time

Compound interest for fixed deposits at regular intervals of time

At the end of	Value of deposit
Year 1	$P_0(1+i) + A_0$
Year 2	$P_0(1+i)^2 + A_0(1+i) + A_0$
Year 3	$P_0(1+i)^3 + A_0(1+i)^2 + A_0(1+i) + A_0$
$\vdots$	$\vdots$
Year t	$P_0(1+i)^t + A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + \cdots + A_0(1+i) + A_0$
Value at the end of t years	$\underbrace{P_0(1+i)^t}_{P_0 \text{ at } i\% \text{ per annum for } t \text{ years}} + \underbrace{A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + \cdots + A_0(1+i) + A_0}_{A_0 \text{ at } i\% \text{ per annum for years } (t-1), (t-2), \dots, 2, 1, 0} \quad (5.20)$

Compound interest for fixed deposits at regular intervals of time

The first term, $P_0(1 + i)^t$ is simply the value of P_0 after t years with an interest rate of $i\%$ per annum compounded annually. The series in the square brackets is the sum of the sequence of deposits, A_0 , made at the end of years $(t-1), (t-2), \dots, 2, 1, 0$ with interest rate $i\%$ per annum compounded annually. This series is actually a geometric series, the sum of which may be calculated according to equation (5.4) or (5.5).

Compound interest for fixed deposits at regular intervals of time

To show how this is accomplished, rewrite the series

$$A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + \cdots + A_0(1+i) + A_0$$

in reverse order and compare this series with the standard geometric series.

$$\underbrace{A_0}_a + \underbrace{A_0(1+i)}_{ar} + \cdots + \underbrace{A_0(1+i)^{t-2}}_{ar^{t-2}} + \underbrace{A_0(1+i)^{t-1}}_{ar^{t-1}} \quad (5.21)$$

Compound interest for fixed deposits at regular intervals of time

The series (5.21) is a geometric series where $a = A_0$ and $r = (1 + i)$. The sum of the first n terms of the standard geometric series is given by (5.5):

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

hence the sum of the first t terms of the series (5.21) is

$$S_t = \frac{A_0[(1 + i)^t - 1]}{(1 + i) - 1} = A_0 \frac{(1 + i)^t - 1}{i} \quad (5.22)$$

$$A_0 = a$$
$$(1 + i) = r$$

Compound interest for fixed deposits at regular intervals of time

Finally the value (V_t) of the investment at the end of t years is equal to initial investment (P_0) plus the annual investments (A_0) all compounded annually:

$$V_t = P_0(1 + i)^t + A_0 \frac{(1 + i)^t - 1}{i} \quad (5.23)$$

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Compound interest for fixed deposits at regular intervals of time

Example:

٥٢ New members of a club are admitted at the start of each year and pay a joining fee of £2000. Henceforth members pay the annual subscription of £400, which falls due at the end of each year. How much does the club earn from a new member over the first 10 years, assuming an annual interest rate of 5.5%.

A_0

$i = 0.055$

t

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Compound interest for fixed deposits at regular intervals of time

Solution:

$$\begin{aligned} P_0 &= 2000, & A_0 &= 400, & i &= \frac{5.5}{100} = 0.055, & t &= 10 \\ V_t &= P_0 (1+i)^t + A_0 \frac{(1+i)^t - 1}{i} \\ &= 2000 (1.055)^{10} + 400 \frac{(1.055)^{10} - 1}{0.055} \\ &= 3416.2889 + 400 \frac{0.7081}{0.055} \end{aligned}$$

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Compound interest for fixed deposits at regular intervals of time

Solution:

$$\begin{aligned} &= 3416.2889 + 400(12.8745) \\ &= \text{£}8566.0889 \end{aligned}$$

Annuities :

An annuity is a series of equal deposits (or withdrawals) made at equal intervals of time; for example, a deposit of £2500 made each year for 20 years towards a pension fund. If the deposit is made at the time of compounding, then the annuity is called an ordinary annuity.

Annuities :

To derive the formula for calculating the value of an annuity at the end of t years, consider its value at the end of each year. as follows.

At the end of	Value of annuity
Year 1	A_0
Year 2	$A_0(1+i) + A_0$
Year 3	$A_0(1+i)^2 + A_0(1+i) + A_0$
Year 4	$A_0(1+i)^3 + A_0(1+i)^2 + A_0(1+i) + A_0$
$\vdots$	$\vdots$
Year t	$A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + A_0(1+i)^{t-3} + \dots + A_0(1+i) + A_0$
Value of annuity after t years	$A_0(1+i)^{t-1} + A_0(1+i)^{t-2} + A_0(1+i)^{t-3} + \dots + A_0(1+i) + A_0$



Annuities :

This series is identical to the geometric series in equation (5.21). Hence the total amount of the annuity at the end of t years is

$$V_{ANU,t} = A_0 \frac{(1+i)^t - 1}{i} \quad (5.24)$$

Note: This result may be deduced as a special case of (5.23) where $P_0 = 0$.

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Thank you

Thank you