

Mathematics 3 – MAT132

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Lecture 1 - Differential Equations: Part 1



Lecture Objective

- To compute derivatives by using the limit definition.
- To develop basic differentiation rules.
- To interpret the derivative as an instantaneous rate of change.
- To apply the product and quotient rules.
- To apply the chain rule.

Lecture Outline



The Derivative

Rules for Differentiation

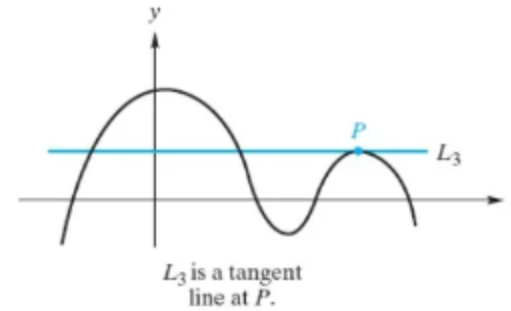
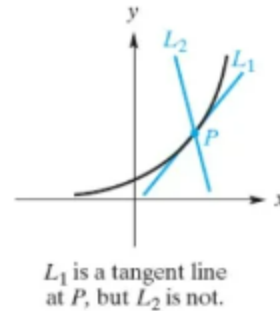
The Product Rule and the Quotient Rule

The Chain Rule and the Power Rule



Derivative

- Tangent line at a point:



- The **slope of a curve** at P is the slope of the tangent line at P .
- The slope of the tangent line at $(a, f(a))$ is

$$m_{\tan} = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

Example

Example 1 – Finding the Slope of a Tangent Line

Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point $(1, 1)$.

Solution:

$$\begin{aligned}\text{Slope} &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h)^2 - (1)^2}{h} = 2\end{aligned}$$

- The **derivative** of a function f is the function denoted f' and defined by

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example

If $f(x) = 2x^2 + 2x + 3$, find an equation of the tangent line to the graph of f at $(1, 7)$.

Solution:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h)^2 + 2(x+h) + 3 - (2x^2 + 2x + 3)}{h} = 4x + 2$$

$$\text{Slope} \rightarrow f'(1) = 4(1) + 2 = 6$$

Derivative

Derivative can take the form:

- $f'(x)$
- $\frac{d}{dx}$

Rule of Differentiation

- Rules for Differentiation:

RULE 1 Derivative of a Constant:

$$\frac{d}{dx}(c) = 0$$

RULE 2 Derivative of x^n :

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

RULE 3 Constant Factor Rule:

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

RULE 4 Sum or Difference Rule

$$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$$



Example 1

Example – A Function with a Vertical Tangent Line

Find $\frac{d}{dx}(\sqrt{x})$.

Solution: $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

Example 2

Example 1 – Derivatives of Constant Functions

a. $\frac{d}{dx}(3) = 0$

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$.

c. If $s(t) = (1,938,623)^{807.4}$, then $\frac{ds}{dt} = 0$.

Solution of Example 2

Example – Rewriting Functions in the Form x^n

Differentiate the following functions:

Solution:

a. $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2} x^{(1/2)-1}$$

b. $h(x) = \frac{1}{x\sqrt{x}}$

$$h'(x) = \frac{d}{dx} (x^{-3/2}) = -\frac{3}{2} x^{(-3/2)-1} = -\frac{3}{2} x^{-5/2}$$

Example

Example – Differentiating Sums and Differences of Functions

Differentiate the following functions:

a. $F(x) = 3x^5 + \sqrt{x}$

$$\begin{aligned} F'(x) &= \frac{d}{dx}(3x^5) + \frac{d}{dx}(x^{1/2}) \\ &= 3(5x^4) + \frac{1}{2}(x^{-1/2}) \end{aligned}$$

b. $f(z) = \frac{z^4}{4} - \frac{5}{z^{1/3}}$

$$\begin{aligned} f'(z) &= \frac{d}{dz}\left(\frac{z^4}{4}\right) - \frac{d}{dz}\left(\frac{5}{z^{1/3}}\right) \\ &= \frac{1}{4}(4z^3) - 5\left(-\frac{1}{3}z^{-4/3}\right) = z^3 + \frac{5}{3}z^{-4/3} \end{aligned}$$

Example

c. $y = 6x^3 - 2x^2 + 7x - 8$

$$\frac{dy}{dx} = 6 \frac{d}{dx}(x^3) - 2 \frac{d}{dx}(x^2) + 7 \frac{d}{dx}(x) - \frac{d}{dx}(8)$$

$$= 18x^2 - 4x + 7$$

The product Rule and Quotient Rule

The Product Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Example

Example 1 – Applying the Product Rule

Find $F'(x)$.

$$F(x) = (x^2 + 3x)(4x + 5)$$

$$\begin{aligned} F'(x) &= \left(\frac{d}{dx}(x^2 + 3x) \right) (4x + 5) + (x^2 + 3x) \left(\frac{d}{dx}(4x + 5) \right) \\ &= (2x + 3)(4x + 5) + (x^2 + 3x)(4) = 12x^2 + 34x + 15 \end{aligned}$$

Example

Example – Differentiating a Product Factors

$$f(x) = (3x - 1)(7x + 2)$$

$$f'(x) = (3)(7x + 2) + (7)(3x - 1)$$

Quotient Rule

The Quotient Rule

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example

Example – Applying the Quotient Rule

$$\text{If } F(x) = \frac{4x^2 + 3}{2x - 1}, \text{ find } F'(x).$$

$$\begin{aligned} \text{Solution: } F'(x) &= \frac{(2x - 1) \frac{d}{dx}(4x^2 + 3) - (4x^2 + 3) \frac{d}{dx}(2x - 1)}{(2x - 1)^2} \\ &= \frac{(2x - 1)(8x) - (4x^2 + 3)(2)}{(2x - 1)^2} = \frac{2(2x + 1)(2x - 3)}{(2x - 1)^2} \end{aligned}$$

Example

Example – Differentiating Quotients without Using the Quotient Rule

Differentiate the following functions.

a. $f(x) = \frac{2x^3}{5}$

$$f'(x) = \frac{2}{5}(3x^2) = \frac{6x^2}{5}$$

Example

$$\text{b. } f(x) = \frac{4}{7x^3} = \frac{4}{7}(x^{-3})$$

$$f'(x) = \frac{4}{7}(-3x^{-4}) = -\frac{12}{7x^4}$$

Example

Example – Differentiating Quotients without Using the Quotient Rule

$$\text{c. } f(x) = \frac{5x^2 - 3x}{4x} = \frac{1}{4}(5x - 3)$$

$$f'(x) = \frac{1}{4}(5) = \frac{5}{4}$$

The chain Rule and Power Rule

The Chain Rule and the Power Rule

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Power Rule: $\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$

Example

Example 1 – Using the Chain Rule

a. If $y = 2u^2 - 3u - 2$ and $u = x^2 + 4$, find dy/dx .

Solution: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{d}{du}(2u^2 - 3u - 2) \cdot \frac{d}{dx}(x^2 + 4)$$

$$= (4u - 3)(2x)$$

$$\frac{dy}{dx} = [4(x^2 + 4) - 3](2x) = [(4x^2 + 13)](2x) = 8x^3 + 26x$$

Example

b. If $y = \sqrt{w}$ and $w = 7 - t^3$, find dy/dt .

$$\begin{aligned}\text{Solution: } \frac{dy}{dt} &= \frac{d}{dw}(w^{1/2}) \cdot \frac{d}{dt}(7 - t^3) \\ &= -\frac{3t^2}{2\sqrt{w}} \\ &= -\frac{3t^2}{2\sqrt{7-t^3}}\end{aligned}$$

Example

Example – Using the Power Rule

If $y = (x^3 - 1)^7$, find y' .

$$\text{Solution: } y' = 7(x^3 - 1)^{7-1} \frac{d}{dx}(x^3 - 1)$$

$$= 7(x^3 - 1)^6 (3x^2)$$

$$= 21x^2(x^3 - 1)^6$$

Example

Example – Using the Power Rule

If $y = \frac{1}{x^2 - 2}$, find dy/dx .

Solution:

$$\begin{aligned}\frac{dy}{dx} &= (-1)(x^2 - 2)^{-1} \frac{d}{dx}(x^2 - 2) \\ &= -\frac{2x}{(x^2 - 2)^2}\end{aligned}$$

Extra Practice

(A) Find $f'(x)$:

1) $f(x) = x^6$

$$f'(x) = 6x^5$$

Extra Practice

(A) Find $f'(x)$:

$$2) f(x) = 4x^3$$

$$f'(x) = 12x^2$$

Extra Practice

(A) Find $f'(x)$:

$$3) f(x) = 4x^2 - 2x + 3$$

$$f'(x) = 8x - 2$$

Extra Practice

$$4) f(x) = 11\sqrt{x}$$

$$f'(x) = 11x^{\frac{1}{2}} = \frac{11}{2}x^{\frac{-1}{2}} = \frac{11}{2\sqrt{x}}$$

Extra Practice

$$5) f(x) = \frac{8}{x^5}$$

$$f'(x) = 8x^{-5} = -40x^{-6}$$

Extra Practice

Find $f'(x)$ by using the product rule and the quotient rule:

$$1) f(x) = (4x + 1)(6x + 3)$$

$$f'(x) = (4)(6x + 3) + (6)(4x + 1)$$

Extra Practice

$$2) f(x) = \frac{2x^2+5x-2}{3x^2+5x+3}$$

$$f'(x) = \left(\frac{(3x^2+5x+3)(4x+5) - (2x^2+5x-2)(6x+5)}{(3x^2+5x+3)^2} \right)$$

Extra Practice

$$3) f(x) = \frac{5x}{x-1}$$

$$f'(x) = \left(\frac{(x-1)(5) - (5x)(1)}{(x-1)^2} \right)$$

Extra Practice

Find $\frac{dy}{dx}$:

1. if $y = u^2 - 2u$ and $u = x^2 - x$

$$\frac{dy}{du} = 2u - 2$$

$$\frac{du}{dx} = 2x - 1$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= (2u - 2)(2x - 1) \\ &= (2(x^2 - x) - 2)(2x - 1)\end{aligned}$$

Lecture Summary

The Derivative

Rules for Differentiation

Lecture Summary

The Product Rule

The Quotient Rule

The Chain Rule

The Power Rule