

Mathematics

Lecture 5

MAT 122

Advanced Topics in Matrix Algebra

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- **The inverse of Matrix Definition**
- **The inverse for 2×2 matrix by determinant**
- **Solving Two Equations by Inverse matrix**
- **The inverse for 2×2 matrix by Reduced Echelon Form**

Let A be a square matrix. A matrix B such that $AB=I=BA$ is called the inverse matrix of A and is denoted by A^{-1} .

So if A^{-1} exists, we have $AA^{-1}=I=A^{-1}A$ and the matrix is said to be invertible.

If a matrix has no inverse, then it is said to be **non-invertible**.

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Definition:

If (A) is an $n \times n$ matrix, form the $n \times 2n$ matrix $(A|I)$, and Perform elementary row operations until the first n columns form a reduced matrix.

We will solve the system of linear equations by unit matrix.

$$\textcircled{A^{-1}} A = A A^{-1} \textcircled{= I}$$

Like a reciprocal
in scalar math

Like the number one
in scalar math

Inverse matrix by determinants

The **inverse of a 2x2 matrix**, say A , is a matrix of the same order denoted by A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix of order 2x2.

i.e., $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In general, the inverse of a matrix A is found using the

formula $(\text{adj } A)/(\det A)$, where "adj A " is the "adjoint of A " and "det A " is

the "determinant of A ". But in case of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can

find the inverse directly using the following formula.

Inverse of 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of A Determinant of A Adjoint of A

Note: A^{-1} exists only when $ad - bc \neq 0$

Note: Since $ad - bc$ is in the **denominator**, it should NOT be 0 for A^{-1} to be defined. i.e.,

- If $ad - bc = 0$, then A^{-1} does NOT exist and in this case, we call A to be a singular matrix.
- If $ad - bc \neq 0$, then A^{-1} exists and in this case, we call A to be a **non-singular matrix**.

Let us see the process of finding the determinant and adjoint of a 2×2 matrix in detail here.

The inverse of a 2x2 matrix formula uses the **determinant** of the matrix.

We know that the determinant of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is **$\det(A) = ad$**

- bc. i.e., to find the determinant, we just multiply the elements of each of the two diagonals and subtract (the product of principal diagonal's elements being the **minuend**).

Determinant of a 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\det A = \boxed{ad - bc}$$

- The determinant of $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} = (3 \times 5) - (2 \times 1) = 15 - 2 = 13$.
- The determinant of $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = (1 \times 2) - (-1 \times 0) = 2 - 0 = 2$.

The **adjoint of a matrix** A is the transpose of the cofactor matrix of A . But for finding the adjoint of a 2x2 matrix, we don't need to have the headache of finding cofactors. There is a trick to find the adjoint of a 2x2

matrix. For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the adjoint is $\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. i.e., to

find the adjoint of a matrix,

- Interchange the elements of the principal diagonal.
- Just change (but do NOT interchange) the signs of the elements of the other diagonal.

Adjoint of 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$



Interchange

Change signs

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: The adjoint of a matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$ is $\text{adj } A = \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$.

The **inverse** of any matrix A is found using the formula $A^{-1} = (\text{adj } A) / (\det A)$. We already know how to find the $\text{adj } A$ and $\det A$ for a 2x2 matrix. For

any 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and $\det A = ad - bc$.

Substituting these values in the formula $A^{-1} = (\text{adj } A) / (\det A)$,

- $A^{-1} = [1/(ad - bc)] \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example:

The inverse of a 2x2 matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$ is,

$$\begin{aligned} A^{-1} &= [1/(3 \times 5 - 2 \times 1)] \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix} \\ &= (1/13) \begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5/13 & -2/13 \\ -1/13 & 3/13 \end{bmatrix} \end{aligned}$$

Here are some important points about the inverse of a 2x2 matrix.

- A^{-1} does not exist when $\det A = 0$. i.e., a matrix is invertible only if $\det A \neq 0$.
- If A^{-1} is the inverse of A , then $AA^{-1} = A^{-1}A = I$.
- The inverse of an identity matrix is itself. i.e., $I^{-1} = I$.
- A^{-1} is used to solve a system of equations.

Solving Equations by Inverse matrix

First precinct: 6 arrests last week equally divided between felonies and misdemeanors.

Second precinct: 9 arrests - there were twice as many felonies as the first precinct.

$$\text{1st Precinct : } x_1 + x_2 = 6$$

$$\text{2nd Precinct : } 2x_1 + x_2 = 9$$

Solve the following system by using inverse

$$6x + 5y = 2$$

$$x + y = -3$$

First step: put the coefficients of the variables (x, y) in matrix A and the numbers after the equal sign in matrix B

$$A = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix}$$

Second Step: Obtain the inverse matrix by determinant method

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix}$$

Third Step:

$$\begin{matrix} & A^{-1} & & B \\ \begin{bmatrix} X \\ Y \end{bmatrix} & = & \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix} * & \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 17 \\ -20 \end{bmatrix} \end{matrix}$$

$$\therefore X=17$$

$$Y= -20$$

Reduced Echelon Form

Find the inverse for matrix A using the reduced echelon form

$$A = \begin{bmatrix} 6 & 5 \\ 1 & 1 \end{bmatrix}$$

Add unit matrix X to matrix A

$$\begin{array}{cc} A & I \\ [A|I] = \begin{bmatrix} 6 & 5 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

you have to convert A to I , by adding the two rows, multiplying number by row, subtracting the two rows or multiplying number by row and add or subtract it from the other one.

So, We need to convert number 6 to 1.

So we will multiply row two by 7 and subtract row 1 from it.
And write the second row as it is.

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$$7R_2 - R_1$$
$$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 7 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

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And write the second row as it is.

$$7R_2 - R_1$$
$$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 7 \\ 1 & 1 & 0 & 1 \end{array} \right]$$

Then we need to convert 1 to zero

So $R_1 - R_2$

and row one will be as it is.

$$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 7 \\ 0 & 1 & -1 & 6 \end{array} \right]$$

Then convert ② to ① then $R_1 - 2R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 6 \end{array} \right]$$

After converting the matrix A to a unit matrix.

Then $A^{-1} = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix} \leftarrow$ Inverse matrix

Let's Practice

Find the inverse of

$$\begin{bmatrix} 5 & 6 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det \begin{pmatrix} 5 & -6 \\ -1 & 2 \end{pmatrix}} \begin{pmatrix} 2 & -6 \\ -(-1) & 2 \end{pmatrix}$$

$$\text{Since, } \det \begin{pmatrix} 5 & 6 \\ -1 & 2 \end{pmatrix} = 16$$

Therefore,

$$A^{-1} = \frac{1}{16} \begin{pmatrix} 2 & -6 \\ -(-1) & 5 \end{pmatrix} = \begin{pmatrix} 1/8 & -3/8 \\ 1/16 & 5/16 \end{pmatrix}$$

Find the inverse of

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{(3)(2) - (1)(4)} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{pmatrix} \end{aligned}$$

Find the inverse of

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{(2)(1) - (4)(-3)} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \\ &= \frac{1}{14} \begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix} \end{aligned}$$

$$A^{-1} = \begin{pmatrix} 1/14 & -4/14 \\ 3/14 & 2/14 \end{pmatrix} = \begin{pmatrix} 1/14 & -2/7 \\ 3/14 & 1/7 \end{pmatrix}$$

Find the inverse of

$$A = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

In this case the determinant of the matrix is zero:

$$\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix} = 3 \times 4 - 2 \times 6 = 0$$

Because the determinant is zero the matrix is singular and no inverse exists.

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