

Mathematics

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Lecture 3

جامعة الجميع الذكية

MAT 112
Limits and Continuity



جامعة الجميع الذكية
EVERYONE'S SMART UNIVERSITY

- To study limits and their basic properties.
- To study one-sided limits, infinite limits, and limits at infinity.
- To study continuity and to find points of discontinuity for a function.



Lecture 03

Lecture Outline

Limits

Continuity



- The limit of $f(x)$ as x approaches a is the number L , written as

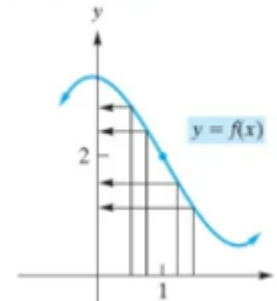
$$\lim_{x \rightarrow a} f(x) = L$$



Example 1 – Estimating a Limit from a Graph

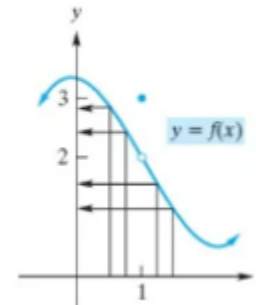
a. Estimate $\lim_{x \rightarrow 1} f(x)$ from the graph.

Solution: $\lim_{x \rightarrow 1} f(x) = 2$



b. Estimate $\lim_{x \rightarrow 1} f(x)$ from the graph.

Solution: $\lim_{x \rightarrow 1} f(x) = 2$



Properties of Limits

1. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c = c$ where c is a constant

2. $\lim_{x \rightarrow a} x^n = a^n$ for any positive integer n

3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

4. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

5. $\lim_{x \rightarrow a} [cf(x)] = c \cdot \lim_{x \rightarrow a} f(x)$



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Properties of Limits

$$6. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$$

$$7. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$



It's very important to substitute first by the value of x , If we have undefined quantity like:

- $\frac{0}{0}$
- $\infty - \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 1^∞
- $\infty \times 0$
- 0^0

Then, we have to use factorization.

Example – Applying Limit Properties 1 and 2

a. $\lim_{x \rightarrow 2} 7 = 7;$ $\lim_{x \rightarrow -5} 7 = 7$

b. $\lim_{x \rightarrow 6} x^2 = 6^2 = 36$

c. $\lim_{t \rightarrow -2} t^4 = (-2)^4 = 16$



Devide any number by infinity ∞ , we will obtain Zero

Devide any number by zero , we will obtain infinity ∞

Example – Limits at Infinity

Find the limit (if it exists)

a. $\lim_{x \rightarrow \infty} \frac{4}{(x-5)^3}$

b. $\lim_{x \rightarrow \infty} \sqrt{4-x}$



Solution:

a. $\lim_{x \rightarrow \infty} \frac{4}{(x-5)^3} = 0$

b. $\lim_{x \rightarrow \infty} \sqrt{4-x} = \infty$



Example – Limit of a Polynomial Function

Find an expression for the polynomial function,

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$$



Solution:

$$\begin{aligned}\lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0) \\&= c_n \lim_{x \rightarrow a} x^n + c_{n-1} \lim_{x \rightarrow a} x^{n-1} + \dots + c_1 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} c_0 \\&= c_n a^n + c_{n-1} a^{n-1} + \dots + c_1 a + c_0 \\&= f(a)\end{aligned}$$

where $\lim_{x \rightarrow a} f(x) = f(a)$



Limits at Infinity for Rational Functions

- If $f(x)$ is a rational function,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_n x^n}{b_m x^m} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{a_n x^n}{b_m x^m}$$



Example – Limits at Infinity for Polynomial Functions

Find the limit (if it exists)

a. $\lim_{x \rightarrow -\infty} (x^3 - x^2 + x - 2)$

b. $\lim_{x \rightarrow \infty} (-2x^3 + 9x)$



Solution: $\lim_{x \rightarrow -\infty} (x^3 - x^2 + x - 2) = -\infty$

Solution: $\lim_{x \rightarrow \infty} (-2x^3 + 9x) = \infty$



Example 1:

To Compute

$$\lim_{x \rightarrow -4} (5x^2 + 8x - 3)$$

Solution:

First, use property 2 to divide the limit into three separate limits. Then use property 1 to bring the constants out of the first two. This gives,

$$\begin{aligned}\lim_{x \rightarrow -4} (5x^2 + 8x - 3) &= \lim_{x \rightarrow -4} (5x^2) + \lim_{x \rightarrow -4} (8x) - \lim_{x \rightarrow -4} (3) \\ &= 5(-4)^2 + 8(-4) - 3 \\ &= 80 - 32 - 3 = 45\end{aligned}$$

Example 2:

To Compute

$$\lim_{x \rightarrow 6} \left[\frac{(x-3)(x-2)}{x-4} \right]$$

Solution:

Given

$$\lim_{x \rightarrow 6} \left[\frac{(x-3)(x-2)}{x-4} \right]$$

$$= \left[\frac{\lim_{x \rightarrow 6} (x-3) \lim_{x \rightarrow 6} (x-2)}{\lim_{x \rightarrow 6} (x-4)} \right]$$

$$= \left[\frac{(6-3)(6-2)}{(6-4)} \right]$$

$$= \left[\frac{(3)(4)}{(2)} \right] = 6$$

It's very important to **substitute** first by the value of x , If we have undefined quantity like:

- $\frac{0}{0}$
- $\infty - \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 1^∞
- $\infty \times 0$
- 0^0

Then, we have to use **Factorization.**

Example 3:

Compute

$$\lim_{x \rightarrow 3} \frac{(x^2 - 9)}{x - 3}$$

Solution: Given

$$\lim_{x \rightarrow 3} \frac{(x^2 - 9)}{x - 3}$$

It is to be noted that, on substituting the value 3 directly to the function, the numerator as well as denominator will become 0, and we know the value

$\frac{0}{0}$, does not exist.

Using the property of squares, we have:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} &= \lim_{x \rightarrow 3} (x + 3) \\ &= 6 \end{aligned}$$



FACTORISATION

$$k(a + b) = k \times a + k \times b$$

Example:

$$3(x + 2y) = 3x + 6y$$

FACTORISATION

$$(a + b)(c + d) = a \times c + a \times d + b \times c + b \times d$$
The diagram illustrates the distributive property for the product of two binomials. It shows the expression (a + b)(c + d) on the left, followed by an equals sign and the expanded form a × c + a × d + b × c + b × d on the right. Four curved arrows originate from the first binomial (a + b) and point to each term in the second binomial (c + d): one from 'a' to 'c', one from 'a' to 'd', one from 'b' to 'c', and one from 'b' to 'd'. This visualizes how each term in the first binomial is multiplied by each term in the second binomial.

Example:

$$(3+x)(x+2y) = 3x + 6y + x^2 + 2xy$$

Factorization by **Common factor**

$$x^2 + 3x = x (x + 3).$$

Example:

$$3x + 9 = 3(x+3).$$

Factorization for Quadratic Polynomial

$$x^2 + (a + b) x + ab = (x+a)(x + b).$$

Example:

$$x^2 + 5x + 6 = (x+2) (x+3)$$

Factorization for the difference between Two squares

$$a^2 - b^2 = (a - b)(a + b)$$

Example:

$$4x^2 - 9 = (2x + 3)(2x - 3)$$

Example:

$$\text{Find } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Solution:

We can't find the limit by substituting $x = 1$ because

$$\frac{(1)^2 - 1}{1 - 1} = \frac{0}{0} \text{ is undefined.}$$

Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares and then cancel out the common term $(x - 1)$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

Evaluate $\lim_{x \rightarrow 2} (8 - 3x + 12x^2)$, if it exists.

Solution

$$\lim_{x \rightarrow 2} (8 - 3x + 12x^2)$$

$$= 8 - 3(2) + 12(4) = \boxed{50}$$



Example – Finding a Limit

Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}$.



$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}$$

$$= \frac{(1)^2 - 1}{1 + 1} = \frac{1 - 1}{2} = 0$$



Solution: $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1}$

$$= \frac{(-1)^2 - 1}{-1 + 1} = \frac{1 - 1}{-1 + 1} = \frac{0}{0} \quad (\text{Undefined Quantity})$$

So, we will use the factorization Method

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} &= \frac{(x + 1)(x - 1)}{(x + 1)} \\ &= \lim_{x \rightarrow -1} (x - 1) = -1 - 1 = -2 \end{aligned}$$

Extra Exercises – Example 8

Evaluate $\lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15}$, if it exists.

Solution:

$$\begin{aligned} \lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15} \\ = \frac{(-5)^2 - 25}{(-5)^2 + 2(-5) - 15} = \frac{0}{0} \quad (\text{Undefined Quantity}) \end{aligned}$$

So, we need to use factorization to obtain the value of limit

$$= \lim_{x \rightarrow -5} \frac{(x - 5)(x + 5)}{(x - 3)(x + 5)}$$

$$= \lim_{x \rightarrow -5} \frac{x - 5}{x - 3} = \boxed{\frac{5}{4}}$$

Evaluate $\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1}$, if it exists.

Solution

$$\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1} :$$

$$= \frac{-6}{10} = \boxed{-\frac{3}{5}}$$



Limits and Algebraic Manipulation

- If $f(x) = g(x)$ for all $x \neq a$, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$



Definition

- $f(x)$ is **continuous** if three conditions are met:
 1. $f(x)$ exists
 2. $\lim_{x \rightarrow a} f(x)$ exists
 3. $\lim_{x \rightarrow a} f(x) = f(a)$



Example 1 – Applying the Definition of Continuity

- a. *Show that $f(x) = 5$ is continuous at 7.*
- b. *Show that $g(x) = x^2 - 3$ is continuous at -4 .*

Solution: Since $\lim_{x \rightarrow 7} f(x) = \lim_{x \rightarrow 7} 5 = 5$,

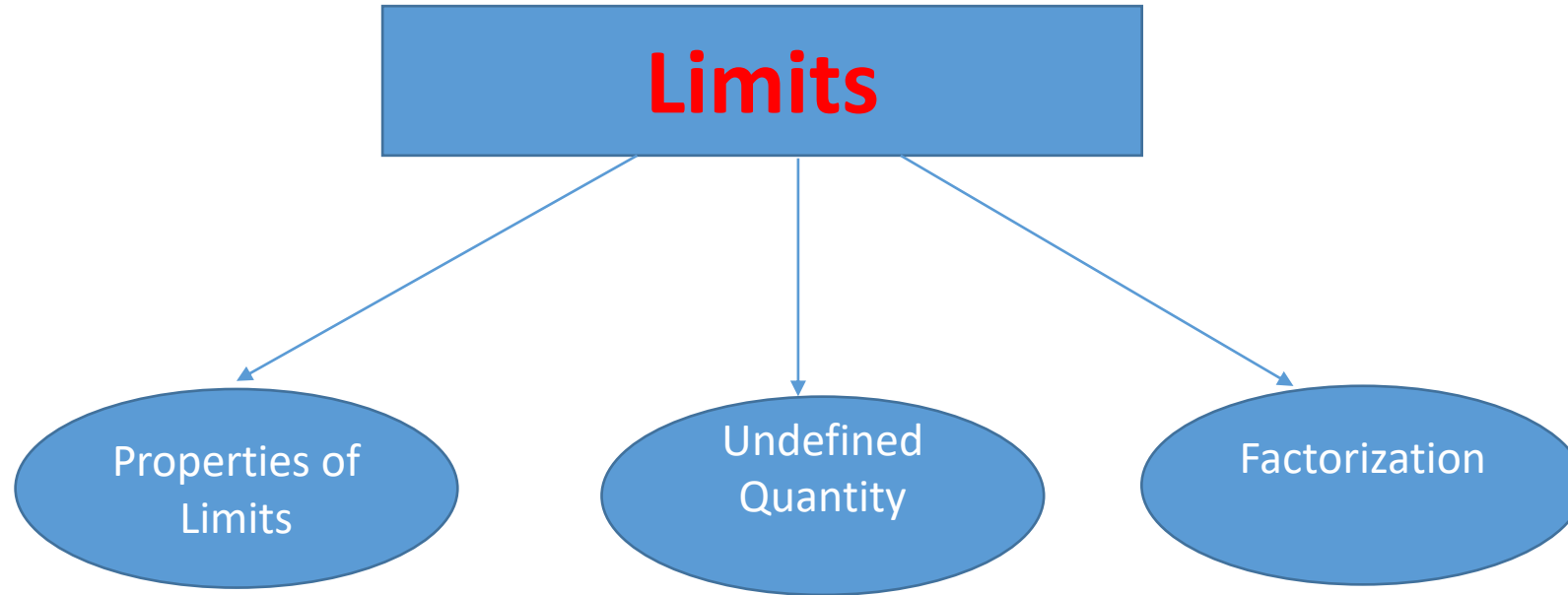
$$\lim_{x \rightarrow 7} f(x) = 5 = f(7).$$

Solution: $\lim_{x \rightarrow -4} g(x) = \lim_{x \rightarrow -4} (x^2 - 3)$

$$= g(-4)$$

$$= (-4)^2 - 3 = 13$$





Continuity

1. $f(x)$ exists
2. $\lim_{x \rightarrow a} f(x)$ exists
3. $\lim_{x \rightarrow a} f(x) = f(a)$

