

Differentiation and Applications

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Lecture 11



جامعة الجميع الذكية
EVERYONE'S SMART UNIVERSITY

MTH 112
Business Mathematics

Differentiation and Applications

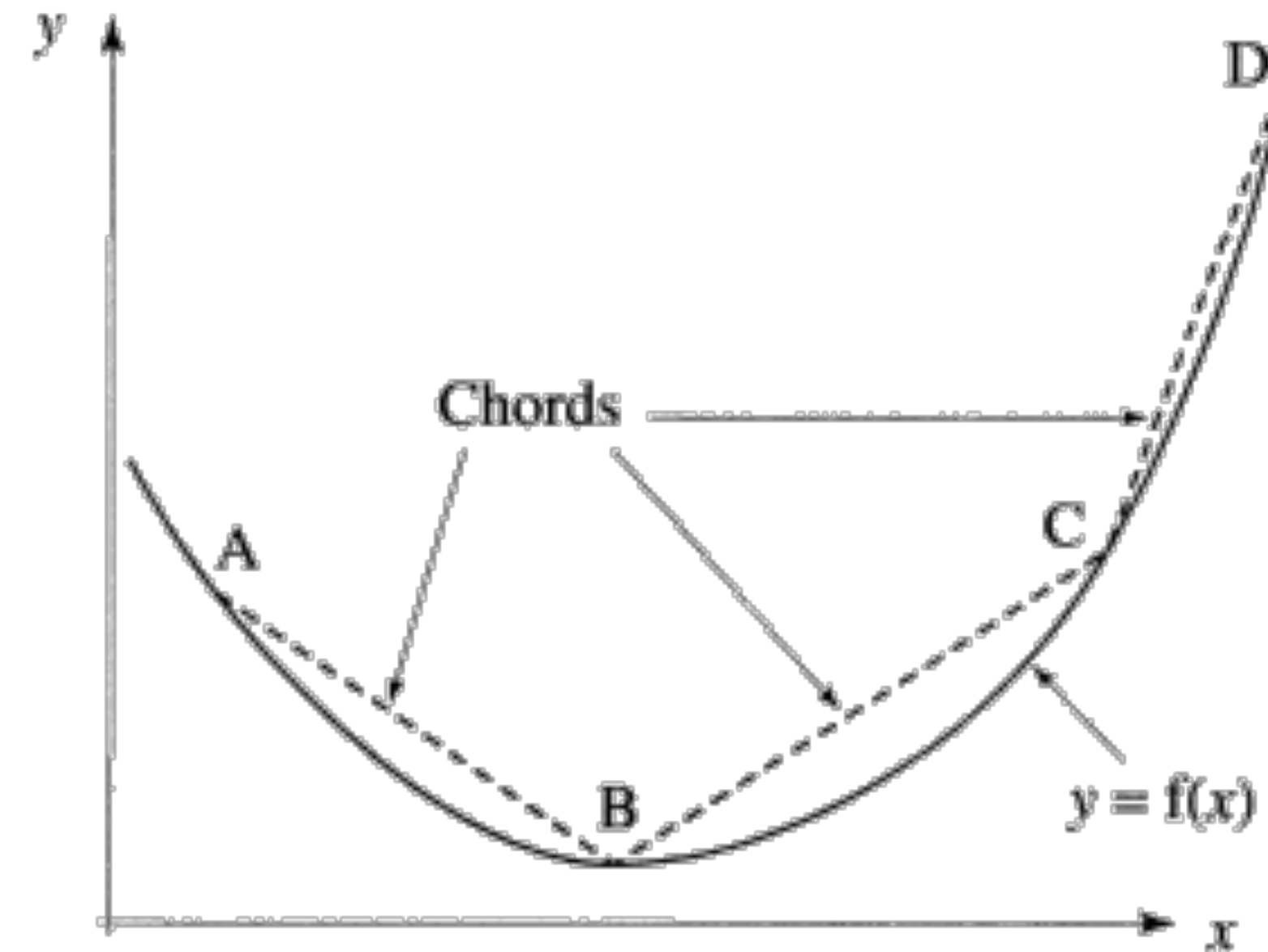
- Slope of a curve and Differentiation
- Optimisation for functions of one variable
- Economic applications of maximum and minimum
- Curvature and other applications

Slope of a Curve and Differentiation

At the end of this section, you should be able to:

- Show how to measure the average slope of a curve between two points
- Show how to calculate the slope of a curve at a point approximately
- Use differentiation to determine the equation for slope from the equation of the curve
- Differentiate functions containing x^n using the power rule
- Use rules for differentiating sums and differences of several functions
- Calculate higher derivatives
- Use differentiation to determine whether a function is increasing or decreasing

The slope of a curve varies at every point on the curve



The slope of the chord BC is a reasonable approximation to the slope of the curve over this interval:

$$\text{Slope of chord } BC = \frac{\text{change in vertical height}}{\text{change in horizontal distance}} = \frac{\Delta y}{\Delta x}$$

The power rule for differentiation

$Y=x^n \rightarrow \frac{dy}{dx} = nx^{n-1}$ is called the power rule

If $y = x^n$ Then $\frac{dy}{dx} = nx^{n-1}$

n drops down

$$\frac{dy}{dx} = nx^{(n-1)} \leftarrow \leftarrow \text{power loses } 1$$

$$y = x^5$$

$$\frac{dy}{dx} = 5x^4$$

$$\text{Slope} = \frac{dy}{dx} = 5x^{5-1} = 5x^4$$



Working rules for differentiating sums and differences of several functions (1/3)

Working rules for differentiating sums and differences of several functions

$$y = x^0$$

$$\frac{dy}{dx} = 0x^{-1}$$

$$= 0$$

Working Rule

Rule 1:

The derivative of a constant term is zero.

Notation

$$y = 10 \rightarrow y = 10x^0$$

$$\therefore \frac{dy}{dx} = 10(0x^{0-1})$$

$$= 0$$

Example

The derivative is the rate of change and the rate of change of a constant term is zero. 10 is a constant term but $10x$ is not a constant term; it varies with x .



Working rules for differentiating sums and differences of several functions (2/3)

$$y = x^n$$

$$\frac{dy}{dx} = nx^{n-1}$$

Working rules for differentiating sums and differences of several functions

Working Rule

Notation

Example

Rule 2:

The derivative of $Kf(x)$, where K is any constant (so $Kf(x)$ is *not* a constant term), is K multiplied by the derivative of $f(x)$.

$$y = Kx^n$$

$$\frac{dy}{dx} = K \frac{d(x^n)}{dx}$$

$$= K(nx^{n-1})$$

$$y = 25x^{0.85}$$

$$\frac{dy}{dx} = 25(0.85x^{0.85-1})$$

$$= 21.25x^{-0.15}$$

$$25 \times 0.85 = 21.25$$



Working rules for differentiating sums and differences of several functions

Working Rule

Rule 3:

To differentiate the sum or difference of several terms, simply differentiate each term separately.

Notation

$$\begin{aligned} y &= K_1 x^m + K_2 x^n \\ \frac{dy}{dx} &= K_1 \{m x^{m-1}\} \\ &\quad + K_2 \{n x^{n-1}\} \\ &= K_1 m x^{m-1} \\ &\quad + K_2 n x^{n-1} \end{aligned}$$

Example

$$\begin{aligned} TC &= 0.5Q^3 - 12Q^2 + 300 \\ \frac{d(TC)}{dQ} &= 0.5(3Q^2) - 12(2Q) + 0 \\ &= 1.5Q^2 - 24Q \end{aligned}$$



Higher derivatives

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = y''$$

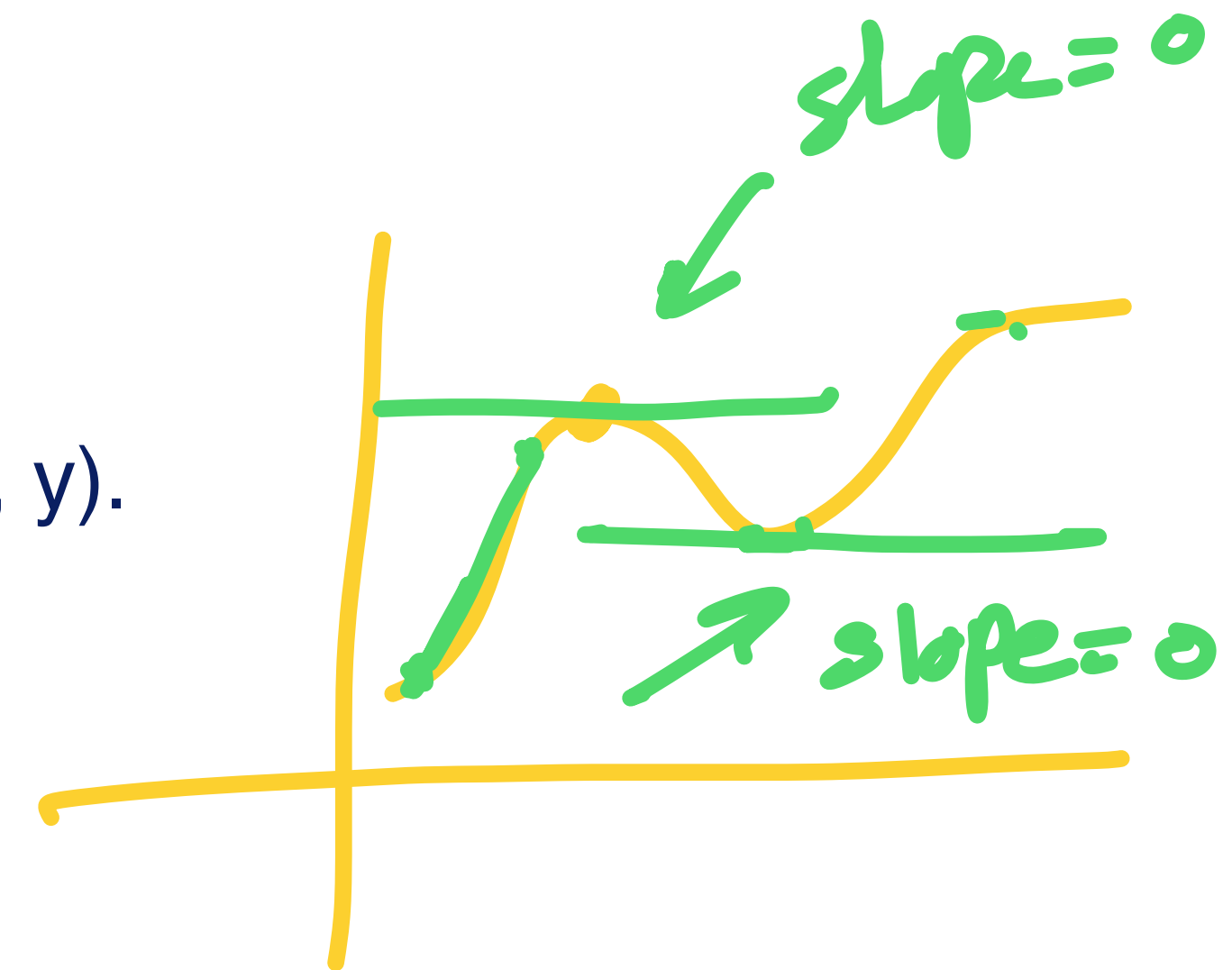
Optimisation for Functions of One Variable

At the end of this section, you should be able to:

- Use the slope of a curve to find turning points
- Use the change in slope and second derivatives to determine whether a turning point is a maximum or minimum
- Find the intervals along which a function is increasing or decreasing
- Outline the key features of curve sketching.

Slope of a curve and turning points

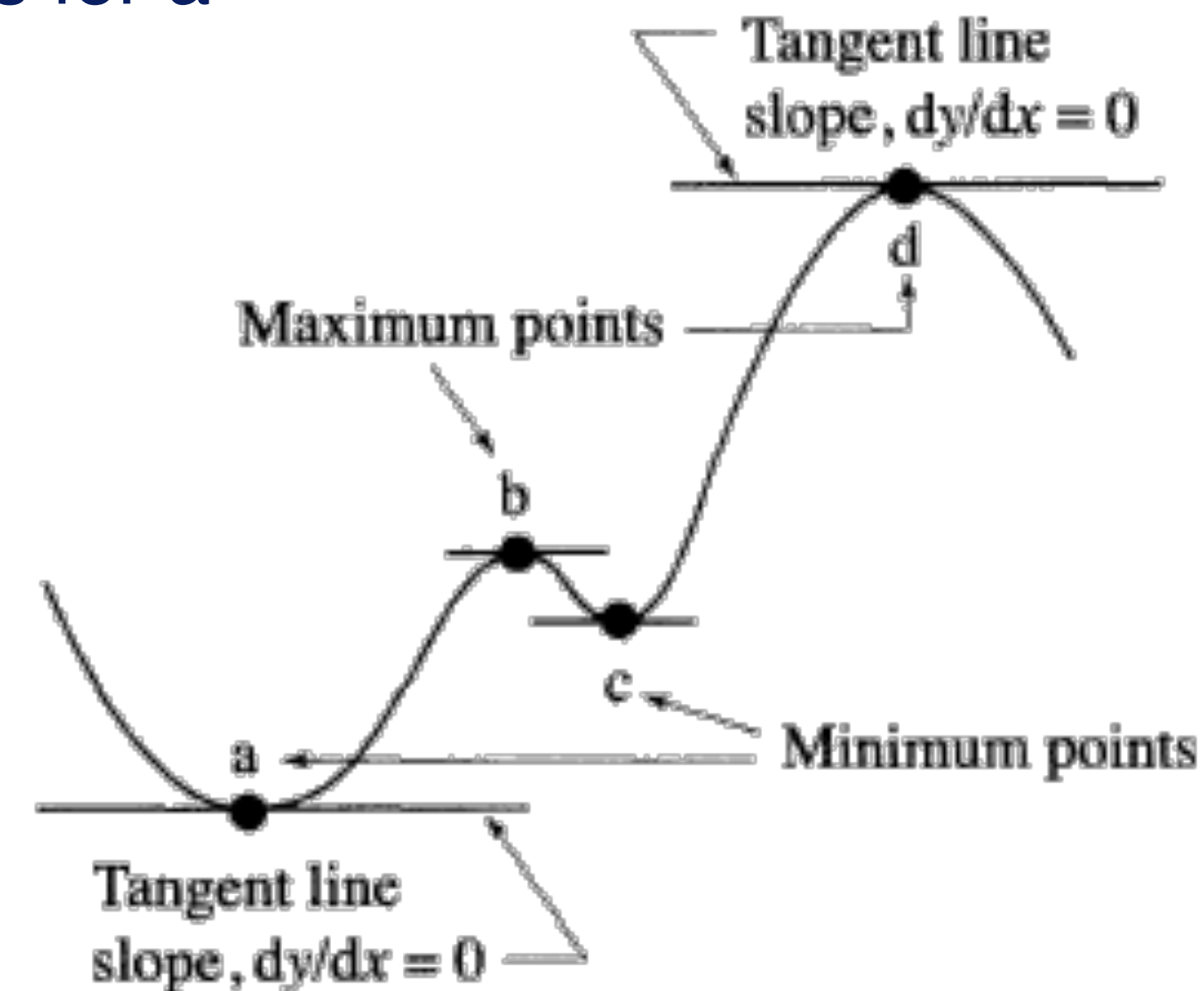
- We know that dy/dx gives the equation for slope at any point (x, y) .
- We know $\text{slope} = 0$ at turning points
- So, $dy/dx = 0$ at turning points.



Therefore, to find the x-coordinates of the turning points for a curve $y = f(x)$, the following method is used:

Step 1: Find dy/dx for the given curve $y = f(x)$.

Step 2: Solve the equation $dy/dx = 0$.



Find the turning points for the following functions

(a) (i) $y = 3x^2 - 18x + 34$
(ii) $AC = 3Q^2 - 18Q + 34$

(b) $y = 1/x$

$$y = 3x^2 - 18x + 34$$

Step 1: $\frac{dy}{dx} = (3)(2)x - (18)(1)x^0 + 0$
 $\text{slope} = 6x - 18$

Step 2: $6x - 18 = 0$
 $6x = 18$
 $x = 3$

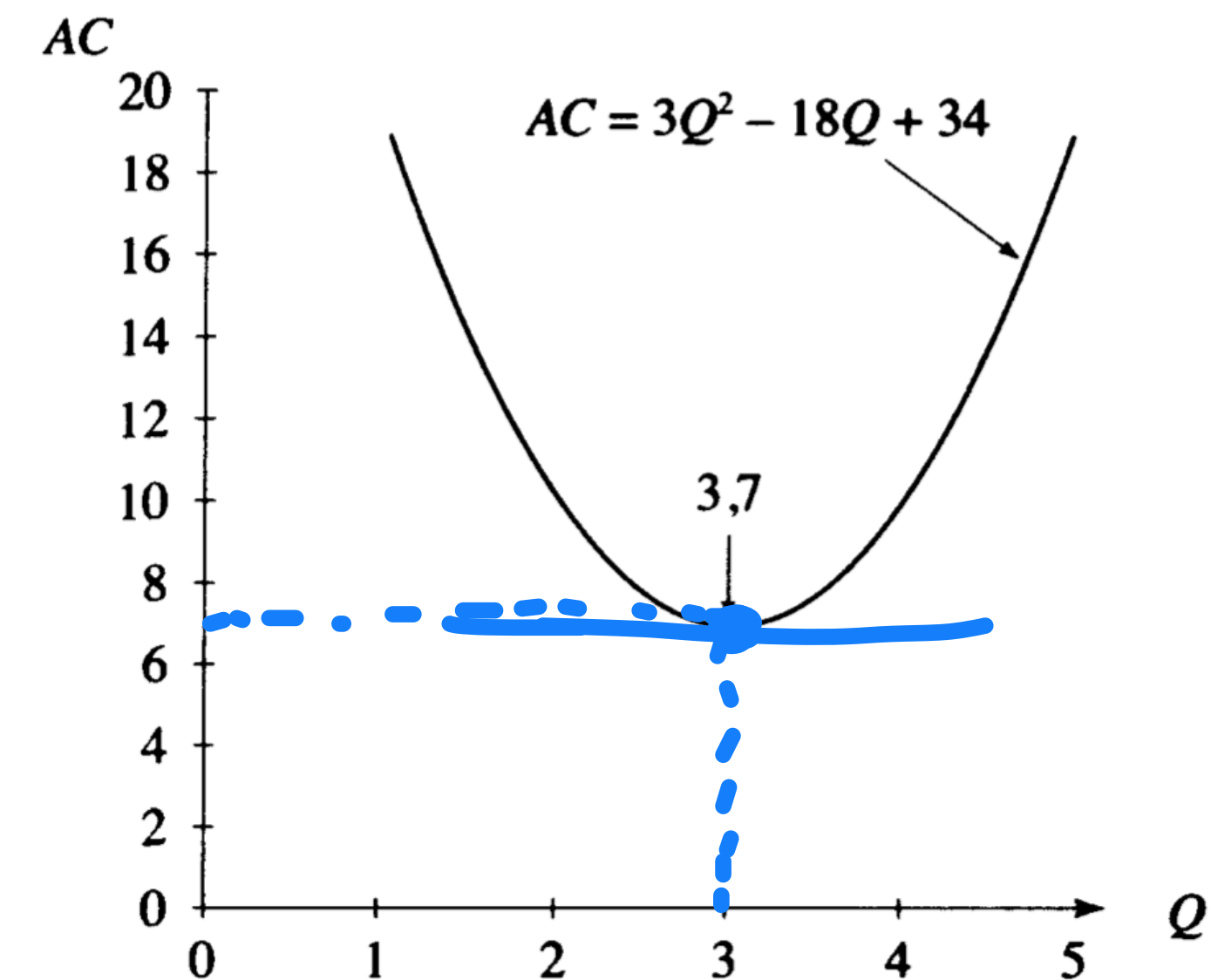


Figure 6.13 Locating turning points

$$y = 3x^2 - 18x + 34$$

$$x = 3$$

$$\begin{aligned} y &= (3)(3)^2 - (18)(3) + 34 \\ &= (3)(9) - 54 + 34 \end{aligned}$$

$$y = 7$$

The turning point exists at $x = 3, y = 7$

$$y = 1/x = x^{-1}$$

$$\text{step 1: } \frac{dy}{dx} = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\text{step 2: } 0 = -\frac{1}{x^2}$$

$$(x^2)(0) = -1$$

$$0 = -x^{-2}$$



There is no value of x^2 when multiplied by 0
will have a value of -1

There is no turning point.



Determining maximum and minimum turning points

The next step is to determine whether each turning point is a maximum or a minimum. We could of course plot the graphs, since we know the precise turning points as in Figure 6.13, but this is time-consuming and not practical in the long term. Before proceeding to a method that will enable us to determine whether a point is a maximum or a minimum, here are some points to note.

When y is increasing then y' is positive
When y' is increasing then y'' is positive

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \quad \text{is Positive}$$

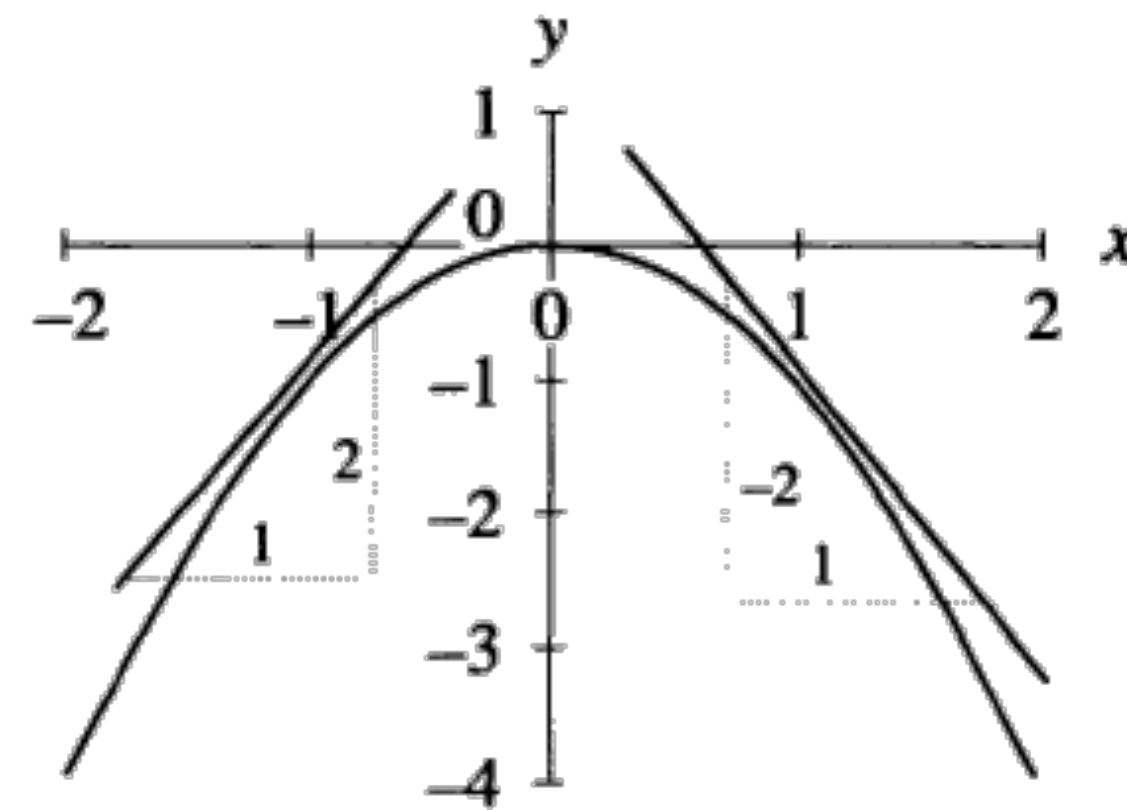
When y is decreasing then y' is negative
When y' is decreasing then y'' is negative

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \quad \text{is Negative}$$

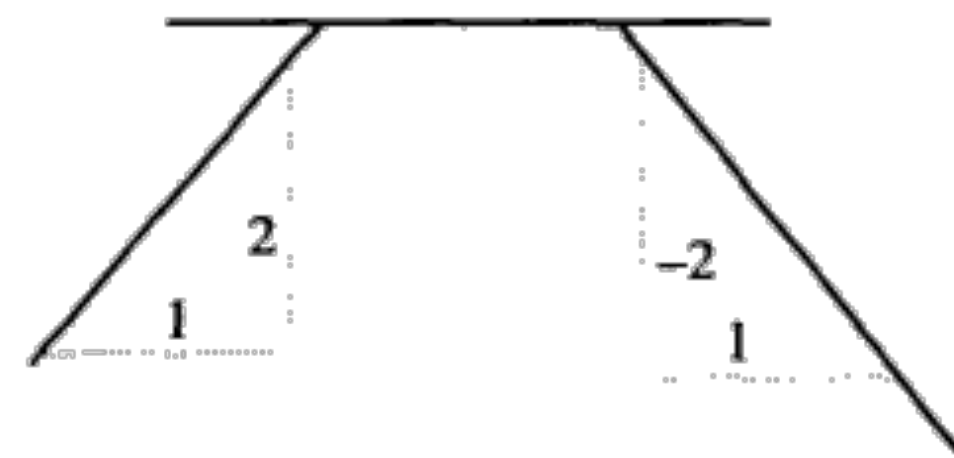
Chapter 06

Testing for minimum and maximum points

MAXIMUM

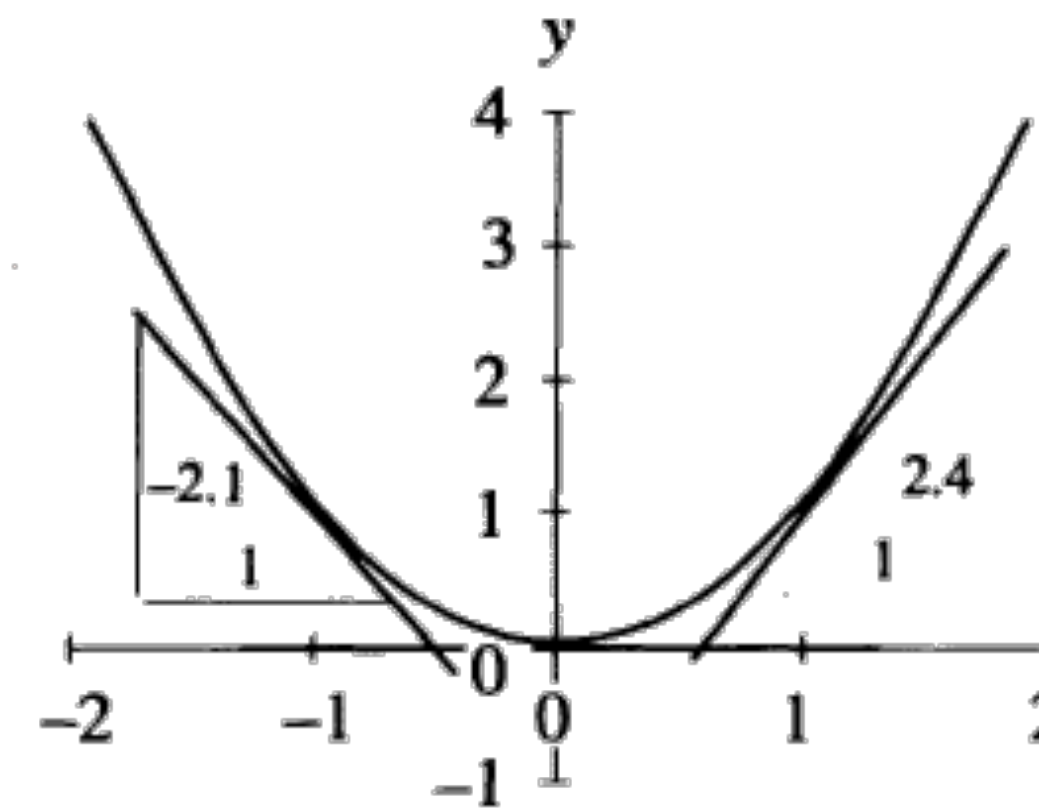


Slopes about a maximum

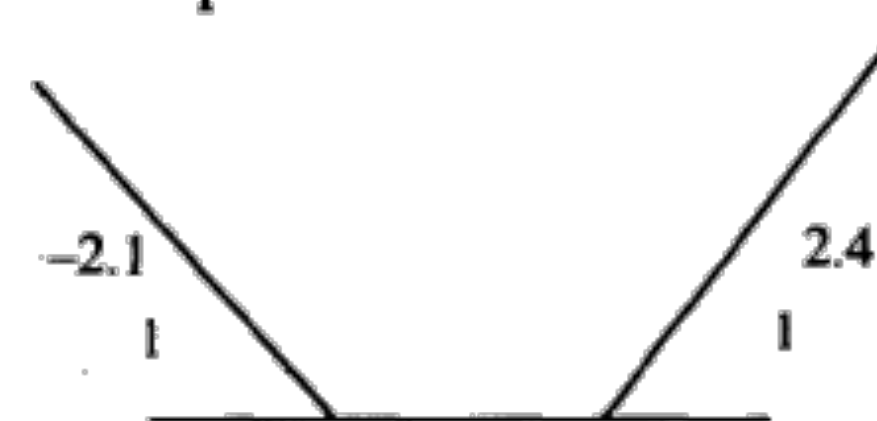


Slope $\rightarrow 2 \rightarrow 0 \rightarrow -2.2$

MINIMUM



Slopes about a minimum



Slope $\rightarrow -2.1 \rightarrow 0 \rightarrow 2.4$

MAXIMUM AND MINIMUM TURNING POINTS

- (a) Find the turning points for the curve, $y = -x^3 + 9x^2 - 24x + 26$.
- (b) Determine which point is a maximum and which is a minimum by using the second derivatives.

$$y = -x^3 + 9x^2 - 24x + 26$$

$$\begin{aligned} \text{slope} = \frac{dy}{dx} &= (-1)(3)x^2 + (9)(2)x - (24) + 0 \\ &= -3x^2 + 18x - 24 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 0 \\ 0 &= -3x^2 + 18x - 24 \\ 0 &= x^2 - 6x + 8 \\ 0 &= (x-2)(x-4) \end{aligned}$$

$$\begin{array}{r} x-4 \\ \times \quad x-2 \\ \hline x^2 - 4x \\ -2x + 8 \\ \hline x^2 - 6x + 8 \end{array}$$

$$\begin{aligned} &ax^2 + bx + c \\ &x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \end{aligned}$$



$$(x-2)(x-4) = 0$$

$$x_1 = 2 \quad / \quad x_2 = 4$$

$$y = -x^3 + 9x^2 - 24x + 26$$

$$y_1 = -(2)^3 + (9)(2)^2 - (24)(2) + 26$$

$$= -8 + 36 - 48 + 26$$

$$y_1 = 6$$

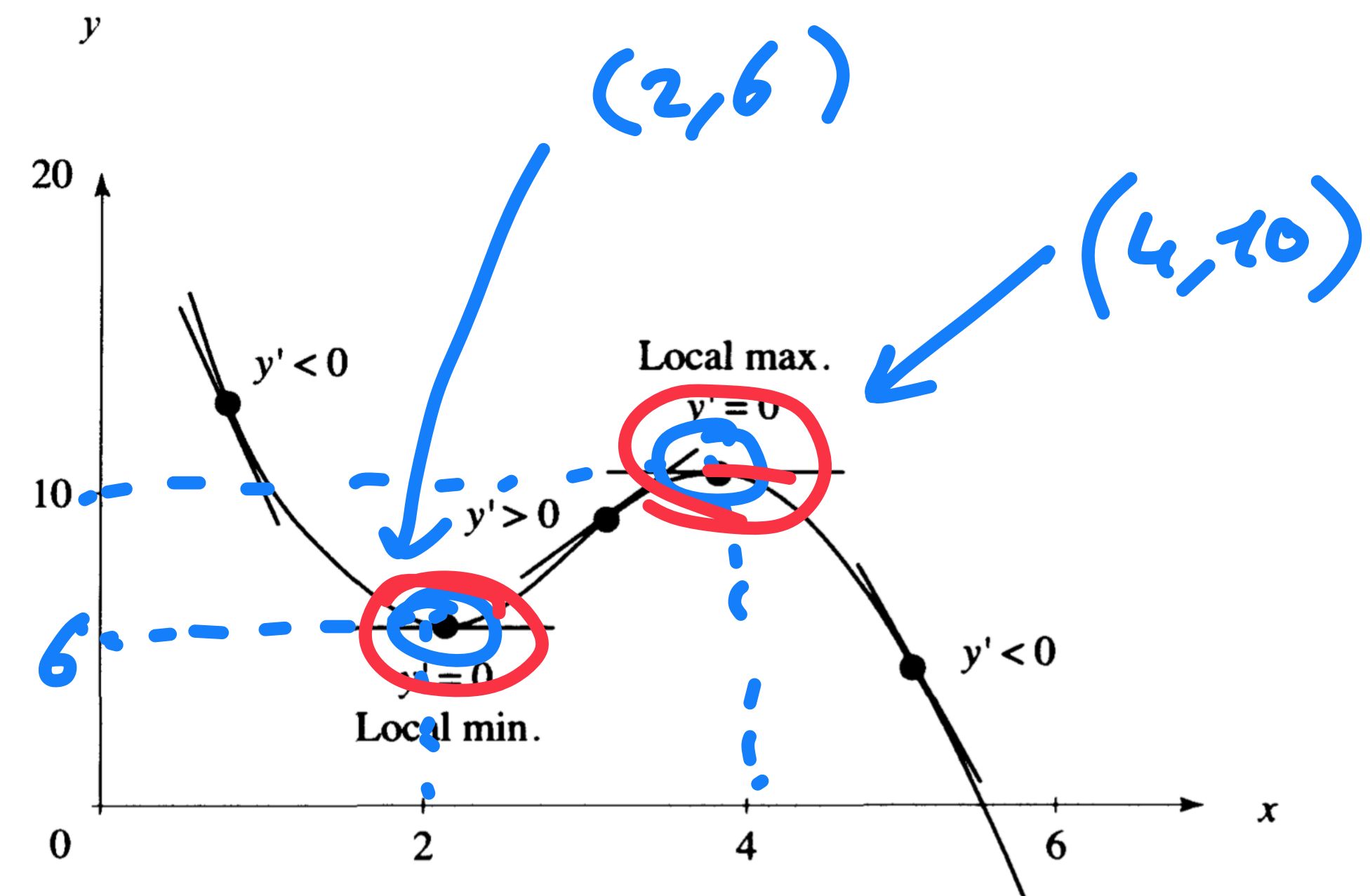


Figure 6.16 Graph of $y = -x^3 + 9x^2 - 24x + 26$

$$y_2 = -(4)^3 + (9)(4)^2 - (24)(4) + 26$$
$$= -64 + 144 - 96 + 26$$

$$\boxed{y_2 = 10}$$

$$\frac{dy}{dx} = -3x^2 + 18x - 24$$

$$\begin{aligned}\frac{d}{dx}\left(\frac{dy}{dx}\right) &= y'' = (-3)(2)x + 18 \\ &= -6x + 18\end{aligned}$$

Turning point
 $x = 2$

$$\begin{aligned}y'' &= -6x + 18 \\&= -6(2) + 18 \\&= 6 \\&\text{Positive}\end{aligned}$$

Therefore, a minimum at $x = 2$

Turning point
 $x = 4$

$$\begin{aligned}y'' &= -6x + 18 \\&= -6(4) + 18 \\&= -6 \\&\text{Negative}\end{aligned}$$

Therefore, a maximum at $x = 4$

$$TR = P \times Q$$

$$\text{Marginal revenue: } MR = \frac{d(TR)}{dQ} \quad (6.5)$$

The marginal revenue is the rate of change in total revenue per unit increase in output, Q :

$$\text{Marginal cost: } MC = \frac{d(TC)}{dQ} \quad (6.6)$$

The marginal cost is the rate of change in total cost per unit increase in output, Q .

MAXIMUM TR AND A SKETCH OF THE TR FUNCTION

The demand function for a good is given as $P = 50 - 2Q$.

- (a) Write down expressions for the TR and MR functions
- (b) Calculate the output at which TR is a maximum and confirm that marginal revenue is zero at this point.
- (c) Sketch the TR and MR functions.

(a)

$$\begin{aligned} TR &= P \times Q \\ &= (50 - 2Q) Q \\ &= 50Q - 2Q^2 \end{aligned}$$

$$MR = \frac{d(TR)}{dQ} = 50 - (2)(2)Q$$
$$MR = 50 - 4Q$$

(b) step 1: $\frac{d(TR)}{dQ} = 50 - 4Q$

step 2: $0 = 50 - 4Q$

$$\cancel{4}Q = \frac{50}{\cancel{4}}$$

$$Q = \frac{50}{4}$$

$$Q = 12.5$$

$$TR = 312.5$$

$$TR = 50Q - 2Q^2$$

$$= (50)(12.5) - (2)(12.5)^2 = 312.5$$

$$TR = 50Q - 2Q^2$$

$$\frac{d(TR)}{dQ} = 50 - 4Q$$

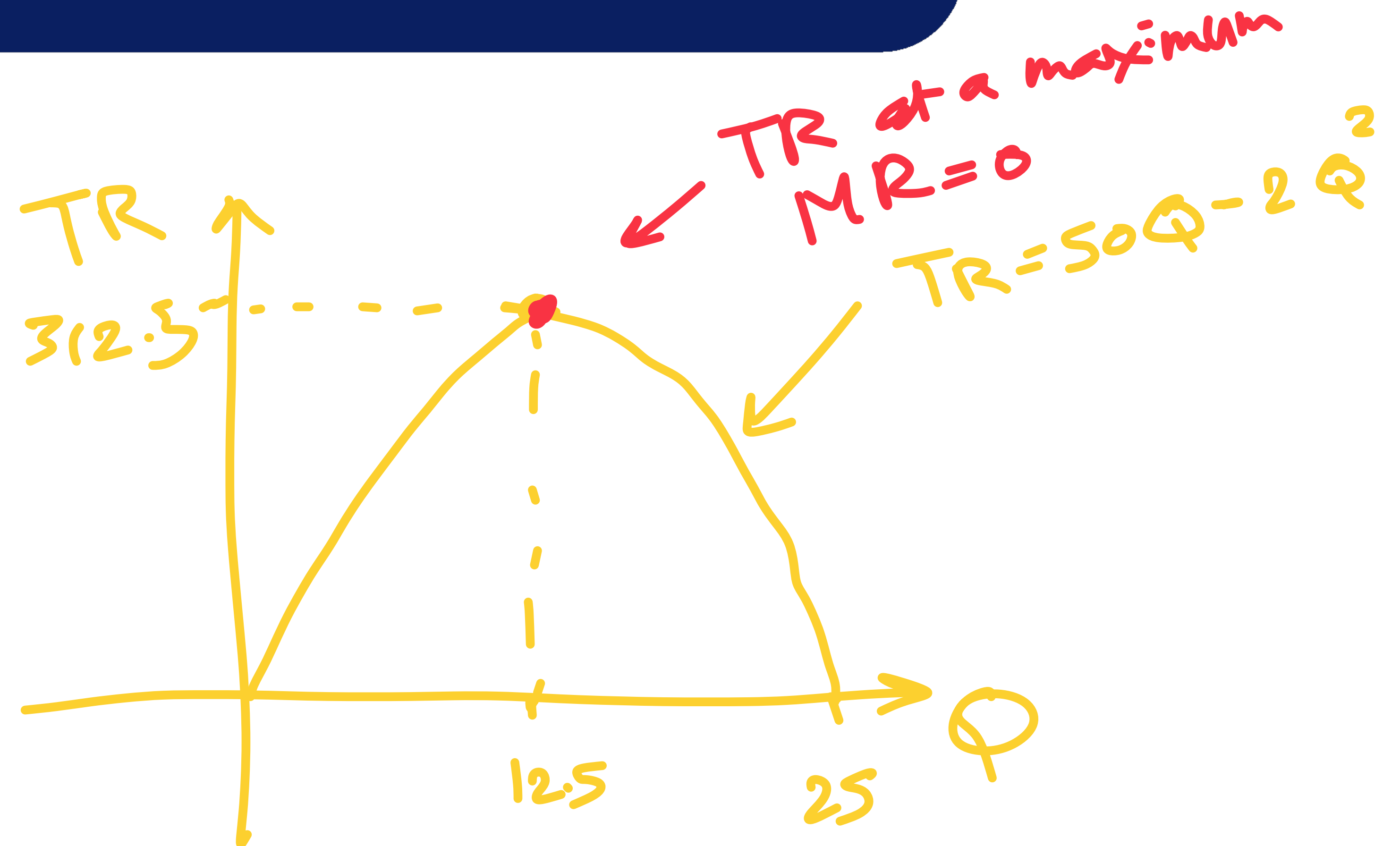
$$y'' = \frac{d^2(TR)}{dQ^2} = -4$$

Negative

The TR is maximised at $TR = 312.5$
 $Q = 12.5$

$$MR = \frac{d(TR)}{dQ} = 50 - 4Q \quad , \quad Q = 12.5$$

$$\begin{aligned} MR &= 50 - (4)(12.5) \\ &= 0 \end{aligned}$$



PROFIT MAXIMISATION AND PRICE DISCRIMINATION

hotel charges different prices for the same meal in two different markets. The demand function in each market is given by the respective equations

$$P_1 = 50 - 4Q_1 \quad \text{and} \quad P_2 = 80 - 3Q_2$$

The company's cost function is given by the equation

$$TC = 120 + 8Q$$

where $Q = Q_1 + Q_2$

- (a) For each market, write down the marginal revenue and marginal cost.
- (b) Determine the price and quantity at which profit is maximized. Calculate the overall profit from the two markets.

Market 1

$$\begin{aligned} \text{(a) } TR &= P_1 \times Q_1 \\ &= (50 - 4Q_1)Q_1 \\ &= 50Q_1 - 4Q_1^2 \end{aligned}$$

$$MR = \frac{d(TR)}{dQ_1} = 50 - 8Q_1$$

Market 2

$$\begin{aligned} TR &= P_2 \times Q_2 \\ TR &= (80 - 3Q_2)Q_2 \\ &= 80Q_2 - 3Q_2^2 \end{aligned}$$

$$MR = \frac{d(TR)}{dQ_2} = 80 - 6Q_2$$

$$(a) \quad MC = \frac{d(TC)}{dQ}$$

$$TC = 120 + 8Q$$

$$MC = \frac{d(TC)}{dQ} = 8$$

(b)

Market 1
 $MR = MC$

$$50 - 8Q_1 = 8$$

$$\frac{50 - 8}{8} = \frac{8Q_1}{8}$$

$$Q_1 = 5.25$$

$$P_1 = 29$$

To maximize

Market 2
 $MR = MC$

$$80 - 6Q_2 = 8$$

$$Q_2 = 12$$

$$P_2 = 44$$



$$\begin{aligned}(MR)' &= -8 \\ (MC)' &= 0 \\ (MR)' &< (MC)'\end{aligned}$$

$$\begin{aligned}(MR)' &= -6 \\ (MC)' &= 0 \\ (MR)' &< (MC)'\end{aligned}$$

Maximum profit

$$\begin{aligned}\pi &= TR - TC \\ &= P_1 Q_1 + P_2 Q_2 - (120 + 8Q) \quad / \quad Q = Q_1 + Q_2 \\ &= (29)(5.25) + (44)(12) - 120 - 8(5.25 + 12) \\ &= 422.25.\end{aligned}$$

Thank you