

Discrete Mathematics

Dr. Shahira Ahmed

IT241

Lecture 6





Discrete Mathematics

Lecture 6

Counting



LEARNING OUTCOMES

- ◉ Solve counting problems by applying elementary counting techniques using the sum and product rules, permutations, combinations and pigeonhole principle.
- ◉ Apply the ideas of Permutations and Combinations, Combinatorial number theory in various real-life situations.



LECTURE OUTLINE

- ◉ The Basics of Counting
- ◉ The Pigeonhole Principle
- ◉ Permutations and Combinations



The Basics of Counting

- Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there?
- Counting problems arise throughout mathematics and computer science. For example, we must count the successful **outcomes** of experiments and **all the possible outcomes** of these experiments to determine probabilities of discrete events.
- We need to **count the number of operations** used by an algorithm to study its time complexity.

The Product Rule:

A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the **first task** and n_2 ways to do the **second task**. Then there are $n_1 * n_2$ **ways** to do the procedure.

Basic Counting Principles: The Product Rule

How many Student code for 2 **digit** number?

How many Student code for 2 **bit** number?

Solution:

$$10 * 10 = 100$$

$$2 * 2 = 4$$

How many **bit** strings of length **seven** are there?

Solution:

Since each of the seven bits is either a 0 or a 1,

$$= 2 * 2 * 2 * 2 * 2 * 2 * 2$$

$$= 2^7$$

$$= 128.$$

How many different license plates can be made if each plate contains a sequence of **three uppercase English** letters followed by **three digits**?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ different possible license plates.

$\underbrace{\quad\quad\quad}_{\substack{26 \text{ choices} \\ \text{for each} \\ \text{letter}}} \quad \underbrace{\quad\quad\quad}_{\substack{10 \text{ choices} \\ \text{for each} \\ \text{digit}}}$

A telephone number consists of 10 digits

- three-digit area code
 - a three-digit office code
 - and a four-digit station code.
- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.

How many different telephone numbers are possible under the old plan and under the new plan?

- $NYX-NXX-XXXX$.
- $NXX-NXX-XXXX$.

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX .
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX .
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format $XXXX$.
- Number of old plan telephone numbers: $160 \cdot 640 \cdot 10,000 = 1,024,000,000$.

- **$NXX-NXX-XXXX$.**

➤ Let X denote a digit from 0 through 9.

➤ Let N denote a digit from 2 through 9.

➤ Let Y denote a digit that is 0 or 1.

Solution: Use the Product Rule.

- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX .
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format $XXXX$.
- **Number of new plan telephone numbers: $800 \cdot 800 \cdot 10,000 = 6,400,000,000$.**
- **Number of old plan telephone numbers: $160 \cdot 640 \cdot 10,000 = 1,024,000,000$.**

$NXX-NXX-XXXX$

➤ Let X denote a digit from 0 through 9.

➤ Let N denote a digit from 2 through 9.

➤ Let Y denote a digit that is 0 or 1.

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 , where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
- Solution
 $23 + 15 + 19 = 57$ ways to choose a project.

The Sum Rule vs The Product Rule

There are **37** faculty member and **83** Student.

1. In how many ways can **one** representatives be picked who is a **either** faculty member or a student?
2. In how many ways can **two** representatives be picked so that one is a faculty member, and the other is a student?

Solution:

1. By the **sum** rule it follows that there $37 + 83 = 120$ possible ways to pick a representative.
2. By the **product** rule it follows that there $37 * 83 = 3071$ possible ways to pick a representative.

Suppose statement Variable name in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels. $\rightarrow$ Letters is upper case

Solution: Use the product rule.

$$26 + 26 \cdot 10 = 286$$

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.
- How many different IPv4 addresses are available for computers on the internet?

Bit Number	0	1	2	3	4	8	16	24	31	
Class A	0	netid				hostid				
Class B	1	0	netid				hostid			
Class C	1	1	0	netid				hostid		
Class D	1	1	1	0	Multicast Address					
Class E	1	1	1	1	0	Address				

- Class **A** addresses, used for the **largest** networks, consist of **0**, followed by a **7-bit** netid and a **24-bit** hostid
- Note :netid 1111111 is unavailable, hostids consisting of all 0s and all 1s are unavailable

Bit Number	0	1	2	3	4	8	16	24	31	
Class A	0	netid					hostid			

- $2^7 - 1 = 127$ netids.
- $2^{24} - 2 = 16,777,214$ hostids.

$$IP = 127 \cdot 16,777,214 = 2,130,706,178.$$

- Class **B** addresses, used for medium-sized networks, consist of **10**, followed by a **14-bit** netid and a **16-bit** hostid.
- Note :hostids consisting of all 0s and all 1s are unavailable

Class B	1	0	netid	hostid
---------	---	---	-------	--------

- $2^{14} = 16,384$ netids
- $2^{16} - 2 = 65,534$ hostids.

$$IP = 65,384 \cdot 65,534 = 1,073,709,056.$$

- Class **C** addresses, used for the smallest networks, a **110**, followed by a **21-bit** netid and a **8-bit** hostid.
- Note :hostids consisting of all 0s and all 1s are unavailable

Class C	1	1	0	netid	hostid
---------	---	---	---	-------	--------

- $2^{21} = 2,097,152$ netids.
- $2^8 - 2 = 254$ hostids.

$$IP = 2,097,152 \cdot 254 = 532,676,608.$$

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31		
Class A	0	netid					hostid				
Class B	1	0	netid					hostid			
Class C	1	1	0	netid					hostid		
Class D	1	1	1	0	Multicast Address						
Class E	1	1	1	1	0	Address					

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- How many different IPv4 addresses are available for computers on the internet?

$$\begin{aligned}x &= x_A + x_B + x_C \\&= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\&= 3,737,091,842\end{aligned}$$

- What is the value of k after the following code, where $n_1, n_2, \dots, n_m$ are positive integers, has been executed?

```
k := 0
for  $i_1 := 1$  to  $n_1$ 
  for  $i_2 := 1$  to  $n_2$ 
    .
    .
    .
    for  $i_m := 1$  to  $n_m$ 
       $k := k + 1$ 
```

- $n_1 * n_2 * \dots * n_m$

- What is the value of k after the following code, where n_1, n_2, n_m are positive integers, has been executed?

```
k := 0
for  $i_1 := 1$  to  $n_1$ 
    k := k + 1
for  $i_2 := 1$  to  $n_2$ 
    k := k + 1
    .
    .
    .
for  $i_m := 1$  to  $n_m$ 
    k := k + 1
```

```
k := 0
for  $i_1 := 1$  to  $n_1$ 
    for  $i_2 := 1$  to  $n_2$ 
        .
        .
        .
        for  $i_m := 1$  to  $n_m$ 
            k := k + 1
```

$$n_1 + n_2 + \dots + n_m$$

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

How many bit strings of length **eight** either **start with a 1** bit or **end with the two bits 00**?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start **with a 1 bit**: $2^7 = 128$

$\overbrace{\text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---}}^{2^7 = 128 \text{ ways}}$
- Number of bit strings of length eight that end **with bits 00**: $2^6 = 64$

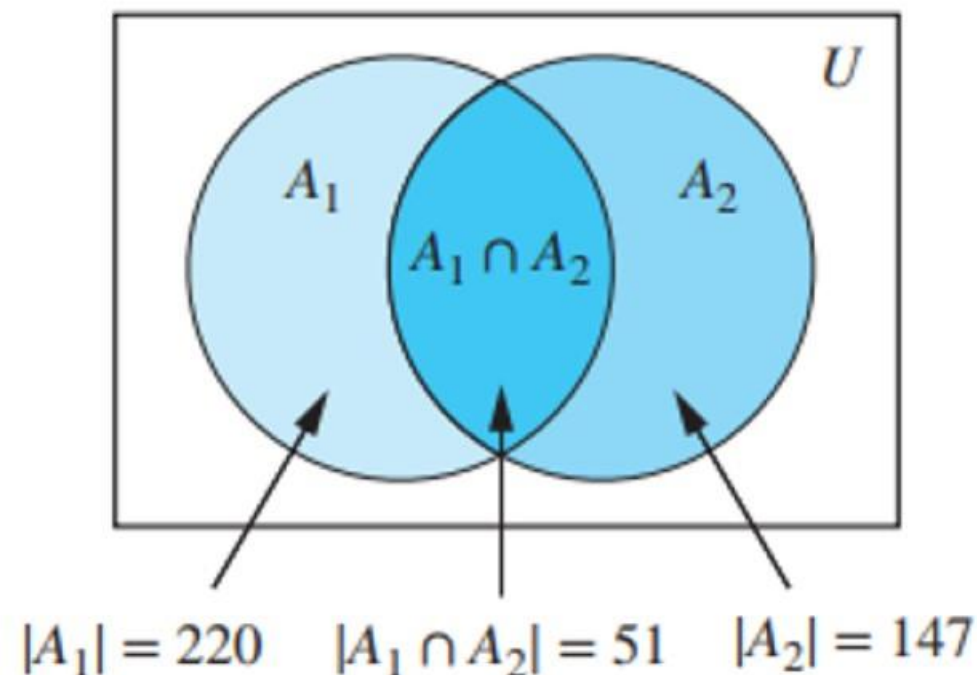
$\overbrace{\text{---} \text{---} \text{---} \text{---} \text{---} \text{---}}^{2^6 = 64 \text{ ways}} \text{---} 0 \text{---} 0$
- Number of bit strings of **length eight that start with a 1 bit** and end with bits 00 : $2^5 = 32$

$\text{---} 1 \overbrace{\text{---} \text{---} \text{---} \text{---} \text{---}}^{2^5 = 32 \text{ ways}} \text{---} 0 \text{---} 0$
- the number is $128 + 64 - 32 = 160$

- Job applications.
 - 220 computer science
 - 147 business
 - 51 computer science and in business,
- How many applicants?

Solution

$$220 + 147 - 51 = 316$$



Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w , exactly d of the n ways correspond to way w .

Suppose 100 cookies are distributed equally to a group of kids. Each kid receives 4 cookies. How many kids are there?

Solution $100/4=25$

The total degree of undirected graph is 58, how many edges does it have?

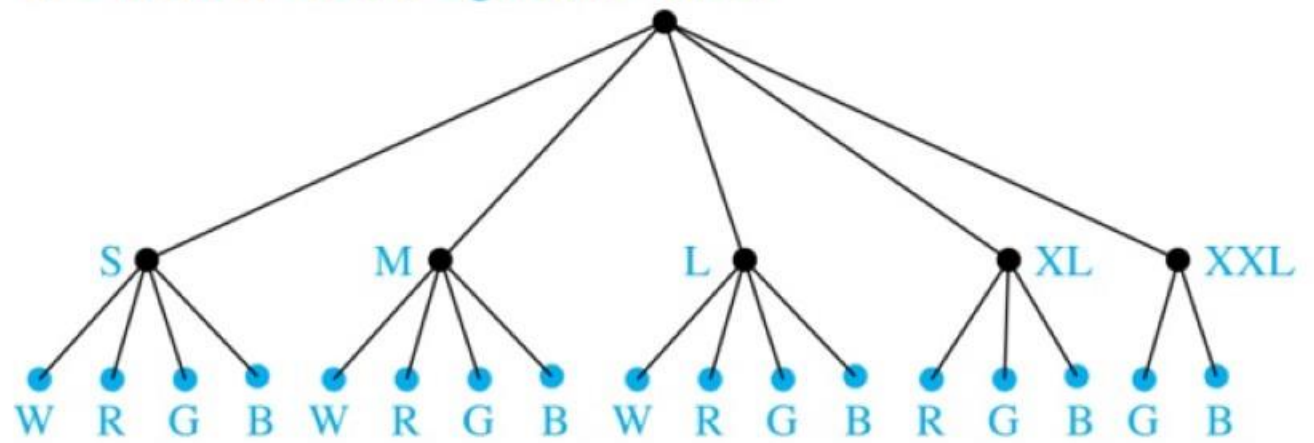
Solution: we count each edge twice, so there are $58/2=29$ edges in the graph

- Suppose that an automated system has been developed that counts the legs of cows in a pasture. Suppose that this system has determined that
 - in a farmer's pasture there are exactly 572 legs.
 - How many cows are there in this pasture, assuming that each cow has four legs and that there are no other animals present?
- Solution:
 - Let n be the number of cow legs counted in a pasture. Because each cow has four legs, by the division rule we know that the pasture contains $n/4$ cows.
 - the pasture with 572 cow legs has $572/4 = 143$ cows in it.

Tree Diagrams

- **Tree Diagrams:** We can solve many counting problems through the use of *tree diagrams*, where a branch represents a possible choice and the leaves represent possible outcomes.
- **Example:** Suppose that “I Love Discrete Math” T-shirts come in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus book store needs to stock to have one of each size and color available?
- **Solution:** Draw the tree diagram.

W = white, R = red, G = green, B = black



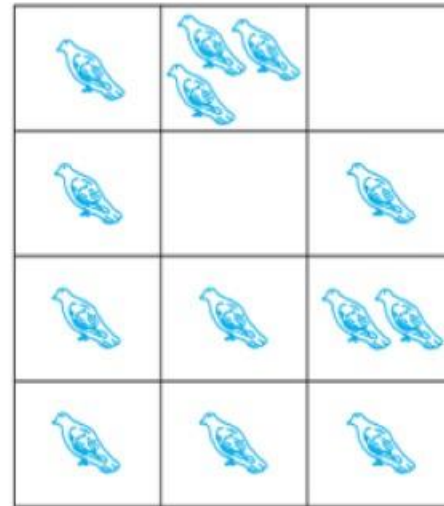
- The store must stock 17 T-shirts.



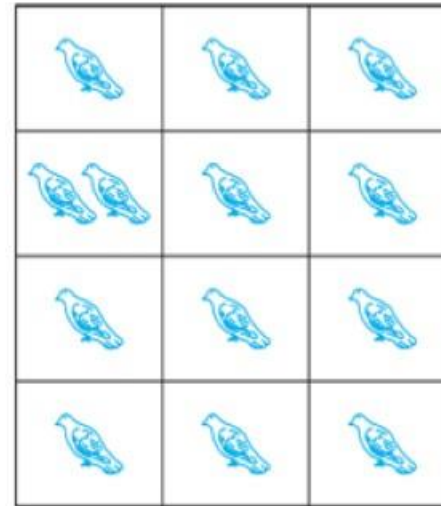
The Pigeonhole Principle

The Pigeonhole Principle

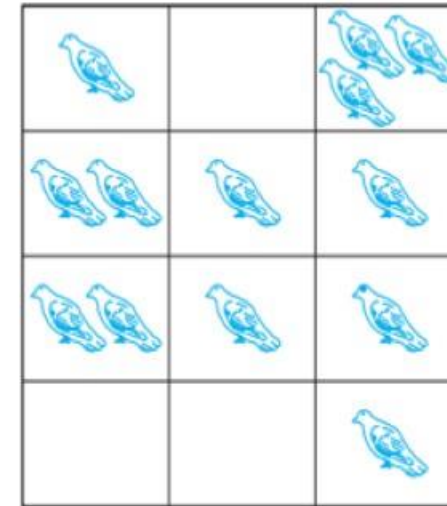
- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



(a)



(b)



(c)

Pigeonhole Principle: If k is a positive integer and $k + 1$ objects are placed into k boxes, then at least one box contains two or more objects.

- Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- In any group of 27 English words, there must be at least two that begin with the same letter.
- How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final.

- What is the minimum number of students required in a discrete mathematics class to be sure that at least one will receive the same grade, if there are five possible grades, A, B, C, D, and F?
- Solution:6.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

1. Ten children are in an elevator. at least ----- of them were born on the same day of the week.
2. Seventeen children are in an elevator. at least -----of them were same day of the week

Solution

1. 2
2. 3



Permutations and Combinations

Definition: A *permutation* of a set of **distinct** objects is an **ordered** arrangement of these objects. An ordered arrangement of r elements of a set is called an *r -permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,2 is a 2-permutation of S .
- The number of r -permutations of a set with n elements is denoted by $P(n,r)$.
 - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2.
 - $P(3,2) = 6$.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Example: Let $S = \{1, 2, 3\}$.

- The ordered arrangement 3,2 is a 2-permutation of S .

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\frac{3!}{1!} = \frac{3*2*1}{1} = 6$$

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$$

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily.

Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

Solving Counting Problems by Counting Permutations

(continued)

Example: How many permutations of the letters $ABCDEFGH$ contain the string ABC ?

Solution: We solve this problem by counting the permutations of six objects, ABC , D , E , F , G , and H .

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Definition: An *r-combination* of elements of a set is an **unordered** selection of *r* elements from the set.

- The number of *r*-combinations of a set with *n* distinct elements is denoted by $C(n, r)$. $\binom{n}{r}$ The notation is also used and is called a *binomial coefficient*.
- The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$,

equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S . It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

How many poker hands of five cards from a standard deck of 52 cards?

Solution:

Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$\begin{aligned} C(52, 5) &= \frac{52!}{5!47!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960 \end{aligned}$$

This is a special case of a general result. →

1. How many poker hands of five cards from a standard deck of 52 cards?
2. How many ways are there to select 47 cards from a deck of 52 cards?

Solution:

1. Since the order in which the cards are dealt does not matter, the number of five card hands is:
$$C(52, 5) = \frac{52!}{5!47!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$
2. The different ways to select 47 cards from 52 is

$$C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960.$$

This is a special case of a general result. →

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution:

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

A group of 30 people. How many ways are there to select six people?

Solution:

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$

The number of r -permutations of a set of n objects with **repetition** allowed is n^r .

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r , which is the number of r -permutations of a set with 26 elements.

The number of r -combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2
is the number of ways to choose six cookies from the four kinds.

$$C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

<i>Type</i>	<i>Repetition Allowed?</i>	<i>Formula</i>
r -permutations	No	$\frac{n!}{(n - r)!}$
r -combinations	No	$\frac{n!}{r! (n - r)!}$
r -permutations	Yes	n^r
r -combinations	Yes	$\frac{(n + r - 1)!}{r! (n - 1)!}$



Recap

There are 18 mathematics majors and 325 computer science majors at a college.

- a) In how many ways can **two** representatives be picked so that one is a mathematics major and the other is a computer science major?
- b) In how many ways can **one** representative be picked who is either a mathematics major or a computer science major?

Solution

- a) $325 * 18 = 5850$
- b) $325 + 18 = 343$

- If a student has an infinite number of red, blue, yellow, and black socks in a drawer, what is the minimum number of socks that the student must pull out of the drawer to guarantee they have a pair?
- Solution 5

A drawer contains **a dozen brown socks** and **a dozen black socks**, all unmatched. A man takes socks out at random in the dark.

- a) How many socks must he take out to be sure that he has at least two socks of the same color?
- Solution
- a) 3

- How many ways are there to select five players from a 10-member tennis team
- Solution

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

- The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
- Solution:
 - The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers.
 - $26 \cdot 100 = 2600$

- Your wardrobe consists of 5 shirts, 3 pairs of pants, and 17 bow ties. How many different outfits can you make?
- Solution
 - There are 255 outfits. Use the multiplicative principle.

- How many strings of **six letters (English-uppercase)** are there?
- Solution: 26^6
- How many ways are there to assign **three** jobs to **five** employees if each employee can be given more than one job?
- Solution: 5^3

- How that in any set of _____ classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day
- Show that if there are _____ students in a class, then at least two have last names that begin with the same letter.
- Solution 8,27

- How many possible Student code for 3 digit number?
- How many possible Student code for 3 bit number?
- Solution: $10^3, 2^3$

- The Product Rule
- The Sum Rule
- The Subtraction Rule
- The Division Rule
- Tree Diagrams
- Permutations with Repetition
- Combinations with Repetition
- Permutations without Repetition
- Combinations without Repetition
- Pigeonhole principle

*Thank
you*

