

Mathematics

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Lecture 6

جامعة الجميع الذكية

تسعى لتكون الكيان الأول الذي يتبنى مبدأ المجتمع التعليمي
التعاوني ويقدم العلم مقابل العمل لتنمية وبناء المجتمعات.

MAT 112

Series: Convergence Tests and Taylor Series



جامعة الجميع الذكية
EVERYONE'S SMART UNIVERSITY

- 1- The first aim is to know well what we mean by infinite series
- 2- Secondly, The definition of convergent and divergent
- 3- It is important to know some notes when solving limits
- 4- Know the different Convergent tests
- 5- Taylor series

- 1- Infinite Series**
- 2- Convergent and Divergent Definition**
- 3- Different Convergent tests**
- 4- Taylor series**

An infinite series is the sum of an infinite sequence of numbers:

$$a_1 + a_2 + \dots + a_n + \dots$$

How can we find this sum of infinite numbers in a finite life time? For his, we look at the sequence of sums of finite number of terms, called the sequence of partial sums:

$$\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$$

The Sum for the first two items is 3

The Sum for the first three items is 6

The Sum for the first 100 items is 5050

The Sum for the first 1000 items is 500500

It is clear that by increasing the number of items, the sum will tend to infinity

So, this series is divergent.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

The Sum for the first two items is 1.5

The Sum for the first three items is 1.75

The Sum for the first 10 items is 1.99805

The Sum for the first 30 items is 1.9999

It is clear that by increasing the number of items, the sum will not be greater than 2

So, this series is Convergent at 2.

1- nth Term Test

2- Direct Comparison Test

3- Limit Comparison Test

4- Cauchy Test

5- Integral Test

6- Ratio Test

7- Root Test

8- Alternating Series Test

we will learn about:

The ways of testing a series for convergence or divergence.

Nth term Test

Limit Comparison Test

Cauchy Test

We will need some important notes for limits

- $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n^2} = e$
- $\lim_{n \rightarrow \infty} \frac{10n+8}{7n+13} = \frac{10}{7}$
- $\lim_{n \rightarrow \infty} \frac{n^3+n^2}{2n^3+3n-4} = \frac{1}{2}$

- $\lim_{n \rightarrow \infty} \frac{\sqrt{3n}}{\sqrt{n-1}} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{33}} \frac{1}{\sqrt{13}} \frac{2\sqrt{3}}{\sqrt{3}} \frac{\sqrt{13}}{\sqrt{3}} \frac{\sqrt{33}}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{2}} \frac{\sqrt{3}}{\sqrt{6}} \frac{\sqrt{3}}{\sqrt{7}}$
- $\lim_{n \rightarrow \infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}} = \left(\frac{2}{4} \right)^2 = \frac{1}{4}$
- $\lim_{n \rightarrow \infty} \left(\frac{5n}{7n-1} \right)^{\frac{3n-1}{n}} = \left(\frac{5}{7} \right)^3 = \frac{125}{343}$

Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

1. converges if $p > 1$, and
2. diverges if $0 < p \leq 1$.

Test Convergent and Divergent of the below P series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Answer: Since $p > 1$, So it is converges

Test Convergent and Divergent of the below P series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Answer: Since $p < 1$, So it is diverges

1- nth Term Test

Limit of n th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Limit of n th-Term Test for Divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Test Convergent and Divergent of the below series using nth term Test

$$\sum_{n=1}^{\infty} \frac{2n + 3}{5n + 11}$$



$$\sum_{n=1}^{\infty} \frac{2n+3}{5n+11} = \frac{5}{16} + \frac{7}{21} + \frac{9}{26} + \dots$$

$$\lim_{n \rightarrow \infty} \frac{2n+3}{5n+11} = \frac{2}{5} \neq 0$$

So, it is Divergence

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = L$$

where L is *finite and positive*. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3 + 3n - 4}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3 + 3n - 4} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{n^2}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{n+1}{2n^3 + 3n - 4} \times \frac{n^2}{1} = \frac{n^3 + n^2}{2n^3 + 3n - 4} = \frac{1}{2} \neq 0$$

The limit is not equal to zero and positive

$$\sum_{n=1}^{\infty} \frac{n+1}{2n^3+3n-4} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ are both convergent or divergent}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ **is** convergent (see Example A)

Then $\sum_{n=1}^{\infty} \frac{n+1}{2n^3+3n-4}$ is convergent

$$\lim_{n \rightarrow \infty} a^n = L$$

$L < 1$ *it will be convergent*

$L > 1$ *it will be divergent*

$L = 1$ *Test failed*

Test Convergent and Divergent of the below series using Cauchy test

$$\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$$



$$\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$$

By using Cauchy test

$$\lim_{n \rightarrow \infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}} = \left(\frac{2}{4} \right)^2 = \frac{1}{4} < 1$$

Then $\sum_{n=1}^{\infty} \left(\frac{2n}{4n-1} \right)^{\frac{2n-1}{n}}$ is convergent

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{n^2}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{1}{n(n+3)} \times \frac{n^2}{1} = \frac{n^2}{n^2 + 3n} = 1 \neq 0$$

The limit is not equal to zero and positive

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ are both convergent or divergent}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (see Example A)

Then $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ is convergent

Test Convergent and Divergent of the below series using Limit Comparison test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}} \quad \longrightarrow \quad \text{Comparison} = \frac{1}{\sqrt{n}}$$

By using comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{3n-1}} \times \frac{\sqrt{n}}{1} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{3n-1}} = \frac{1}{\sqrt{3}} \neq 0$$

The limit is not equal to zero and positive

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ are both convergent or divergent}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (see Example B)

Then $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$ is divergent

Test Convergent and Divergent of the below series using Cauchy test

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$$

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} = 2 + \frac{3^4}{2} + \frac{4^9}{3} + \dots$$

By using Cauchy test

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n^2} = e > 1$$

Then $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$ is divergent

Definitions of n th Taylor Polynomial

If f has n derivatives at c , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **n th Taylor polynomial for f at c .**

Let $f(x) = x^3$

$C = 1$

Find $P_2(x)$ as $n = 2$



$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2$$

as $n = 2$ and $C = 1$

$$P_2(x) = 1 + 3(x - 1) + \frac{6}{2!}(x - 1)^2$$

$$P_2(x) = 1 + 3(x - 1) + \frac{6}{2}(x^2 - 2x + 1)$$

$$P_2(x) = 1 + 3x - 3 + 3(x^2 - 2x + 1)$$

$$P_2(x) = 1 + 3x - 3 + 3x^2 - 6x + 3$$

$F(x) = x^3$	$F(c) = 1^3 = 1$
$f'(x) = 3x^2$	$f'(c) = 3(1)^2 = 3$
$f''(x) = 6x$	$f''(c) = 6(1) = 6$

$$**$P_2(x) = 3x^2 - 3x + 1$**$$



Let $f(x) = 2x^2$

$C = 1$

Find $P_2(x)$ as $n = 2$



$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2$$

as $n = 2$ and $C = 1$

$$P_2(x) = 2 + 4(x - 1) + \frac{4}{2!}(x - 1)^2$$

$$P_2(x) = 2 + 4(x - 1) + \frac{4}{2}(x^2 - 2x + 1)$$

$$P_2(x) = 2 + 4x - 4 + 2(x^2 - 2x + 1)$$

$$P_2(x) = 2 + 4x - 4 + 2x^2 - 4x + 2$$

$$**P_2(x) = 2x^2**$$

$F(x) = 2x^2$	$F(c) = 2 * 1^2 = 2$
$f'(x) = 4x$	$f'(c) = 4 * 1 = 4$
$f''(x) = 4$	$f''(c) = 4$



1- Infinite Series

2- Convergent and Divergent Definition



3- Different Convergent tests

Nth term Test

Limit Comparison Test

Cauchy Test

3- Different Convergent tests

Nth term Test

Limit Comparison Test

Cauchy Test

4- Taylor series

Differential Revision is essential

