

Mathematics

Lecture 9

MAT 112

Multivariable Calculus - Partial derivatives

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- To discuss functions of several variables and to compute function values.
 - To compute partial derivatives.
 - To find partial derivatives of a function defined implicitly.
 - To compute higher-order partial derivatives.
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- To apply the second-derivative test for a function of two variables.
 - To find critical points for a function.
 - To compute double and triple integrals.
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Functions of Several Variables

Partial Derivatives

Implicit differentiation

Maxima and Minima for functions of Two Variable

Functions of Several Variables

- A function can involve 2 or more variables, e.g.

$$z = f(x, y) = \frac{2}{x^2 - y^2}$$

Example 1 – Functions of Two Variables

a. $f(x, y) = \frac{x+3}{y-2}$ is a function of two variables. Because the

denominator is zero when $y = 2$, the domain of f is all (x, y) such that $y \neq 2$.

b. $h(x, y) = 4x$ defines h as a function of x and y . The domain is all ordered pairs of real numbers.

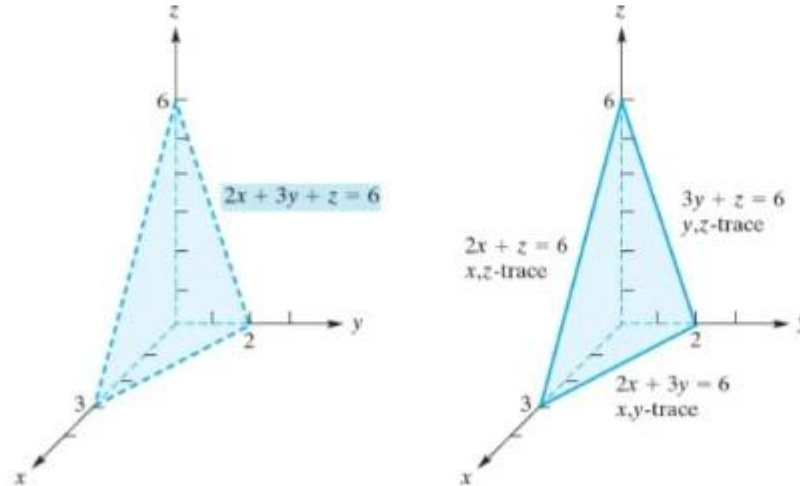
$z^2 = x^2 + y^2$ does not define z as a function of x and y .

Example – Graphing a Plane

Sketch the plane $2x + 3y + z = 6$.

Solution:

The plane intersects the x-axis when $y = 0$ and $z = 0$.

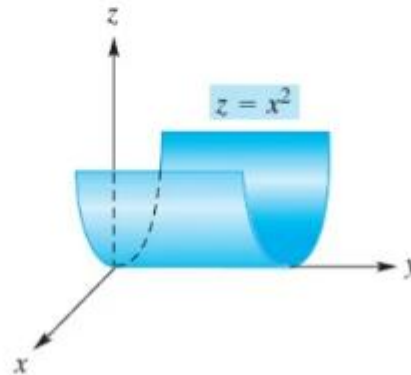


Example – Sketching a Surface

Sketch the surface $z = x^2$.

Solution:

$z = x^2$ is a parabola.



Example 1 – Finding Partial Derivatives

If $f(x, y) = xy^2 + x^2y$, find $f_x(x, y)$ and $f_y(x, y)$. Also, find $f_x(3, 4)$ and $f_y(3, 4)$.

Solution: The partial derivatives are

$$f_x(x, y) = (1)y^2 + (2x)y = y^2 + 2xy$$

$$f_y(x, y) = x(2y) + x^2(1) = 2xy + x^2$$

Thus, the solutions are

$$f_x(3, 4) = 40$$

$$f_y(3, 4) = 33$$

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Example – Partial Derivatives of a Function of Three Variables

If $f(x, y, z) = x^2 + y^2z + z^3$, find $f_x(x, y, z)$, $f_y(x, y, z)$ and $f_z(x, y, z)$.

Solution:

By partial differentiating, we get

$$f_x(x, y, z) = 2x$$

$$f_y(x, y, z) = 2yz$$

$$f_z(x, y, z) = y^2 + 3z^2$$

Implicit Differentiation

Implicit Differentiation Procedure

1. Differentiate both sides.
 2. Collect all dy/dx terms on one side and other terms on the other side.
 3. Factor dy/dx terms.
 4. Solve for dy/dx .
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Example 1 – Implicit Differentiation

Find dy/dx by implicit differentiation if $y + y^3 - x = 7$.

Solution: $\frac{d}{dx}(y + y^3 - x) = \frac{d}{dx}(7)$

$$\frac{dy}{dx} + 3y^2 \frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = \frac{1}{1 + 3y^2}$$

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

$$x + 4yy' = 0$$

$$4yy' = -x$$

$$y' = -\frac{x}{4y}$$

$$3x^2 + 6y^2 = 1$$

$$6x + 12yy' = 0$$

$$y' = -\frac{x}{2y}$$

$$2y^3 - 7x^2 = 5$$

$$6y^2 y' - 14x = 0$$

$$y' = \frac{14x}{6y^2} = \frac{7x}{3y^2}$$

$$5y^2 - 2x^2 = 10$$

$$10yy' - 4x = 0$$

$$y' = \frac{4x}{10y} = \frac{2x}{5y}$$

$$\sqrt[3]{x} + \sqrt[3]{y} = 3$$

$$x^{1/3} + y^{1/3} = 3$$

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}y' = 0$$

$$y^{-2/3}y' = -x^{-2/3}$$

$$y' = -\frac{x^{-2/3}}{y^{-2/3}}$$

$$= -\frac{y^{2/3}}{x^{2/3}} = -\frac{\sqrt[3]{y^2}}{\sqrt[3]{x^2}} = -\sqrt[3]{\frac{y^2}{x^2}}$$

$$x^{\frac{1}{5}} + y^{\frac{1}{5}} = 4$$

$$\left(\frac{1}{5}\right)x^{-\frac{4}{5}} + \left(\frac{1}{5}\right)y^{-\frac{4}{5}}y' = 0$$

$$y' = -\frac{y^{\frac{4}{5}}}{x^{\frac{4}{5}}} = -\left(\frac{y}{x}\right)^{\frac{4}{5}}$$

$$x^{\frac{3}{4}} + y^{\frac{3}{4}} = 5$$

$$\left(\frac{3}{4}\right)x^{-\frac{1}{4}} + \left(\frac{3}{4}\right)y^{-\frac{1}{4}}y' = 0$$

$$y' = -\frac{y^{\frac{1}{4}}}{x^{\frac{1}{4}}}$$

$$y^3 = 4x$$

$$3y^2 y' = 4$$

$$y' = \frac{4}{3y^2}$$

$$y^5 = 7x$$

$$5y^4y' = 7$$

$$y' = \frac{7}{5y^4}$$

$$y^9 = 7x$$

$$9y^8y' = 7$$

$$y' = \frac{7}{9y^8}$$

$$2y^9 = x$$

$$18 y^8 y' = 1$$

$$y' = \frac{1}{18 y^8}$$

$$10y^5 = 9x$$

$$50y^4y' = 9$$

$$y' = \frac{9}{50y^4}$$

$$30 y = x^5$$

$$30 y' = 5x^4$$

$$y' = \frac{5x^4}{30}$$

Maxima and Minima for functions of Two Variables

- **Relative maximum** at the point (a, b) is shown as

$$f(a, b) \geq f(x, y)$$

RULE 1

Find relative maximum or minimum when $\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$;

RULE 2 Second-Derivative Test for Functions of Two Variables

Let D be the function defined by $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, relative maximum at (a, b) ;
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, relative minimum at (a, b) ;
3. If $D(a, b) < 0$, then f has a saddle point at (a, b) ;
4. If $D(a, b) = 0$, no conclusion.

$$y = \frac{1}{3}x^3 + 2x^2 - 5x + 1$$

$$y' = x^2 + 4x - 5 = (x + 5)(x - 1)$$

$$\text{CV: } x = -5, 1$$

$$y'' = 2x + 4$$

$$y''(-5) = -6 < 0 \Rightarrow \text{relative maximum when}$$

$$x = -5$$

$$y''(1) = 6 > 0 \Rightarrow \text{relative minimum when } x = 1$$

$$\underline{y = x^3 - 12x + 1}$$

$$y' = 3x^2 - 12 = 3(x + 2)(x - 2)$$

$$\text{CV: } x = \pm 2$$

$$y'' = 6x$$

$$y''(-2) = -12 < 0 \Rightarrow \text{relative maximum when } x = -2$$

$$y''(2) = 12 > 0 \Rightarrow \text{relative minimum when } x = 2$$

$$y = 2x^3 - 3x^2 - 36x + 17$$

$$y' = 6x^2 - 6x - 36 = 6(x - 3)(x + 2)$$

$$\text{CV: } x = 3, -2$$

$$y'' = 12x - 6$$

$$y''(3) = 30 > 0 \Rightarrow \text{relative minimum when } x = 3$$

$$y''(-2) = -30 < 0 \Rightarrow \text{relative maximum when}$$

$$x = -2$$

$$y = 15x^3 + x^2 - 15x + 2$$

$$y' = 45x^2 + 2x - 15 = (5x + 3)(9x - 5)$$

$$\text{CV: } x = -\frac{3}{5}, \frac{5}{9}$$

$$y'' = 90x + 2$$

$$y''\left(-\frac{3}{5}\right) = -52 \Rightarrow \text{relative maximum when}$$

$$x = -\frac{3}{5}$$

$$y''\left(\frac{5}{9}\right) = 52 \Rightarrow \text{relative minimum when } x = \frac{5}{9}$$

$$\underline{y = -x^3 + 3x^2 + 9x - 2}$$

$$\begin{aligned} y' &= -3x^2 + 6x + 9 = -3(x^2 - 2x - 3) \\ &= -3(x+1)(x-3) \end{aligned}$$

$$\text{CV: } x = -1, 3$$

$$y'' = -6x + 6$$

$$y''(-1) = 12 > 0 \Rightarrow \text{relative minimum when}$$

$$x = -1$$

$$y''(3) = -12 < 0 \Rightarrow \text{relative maximum when}$$

$$x = 3$$

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