

Sequences and Financial mathematics

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Lecture 07



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MTH 112
Business Mathematics

Chapter 05

At the end of this section you should be familiar with:

- Arithmetic and Geometric Sequences and Serieses
- Simple Interest
- Compound Interest
- Present Value
- Future Value

Revision on Arithmetic and Geometric sequences definitions

Arithmetic sequence

If the rule is that each term, after the first, is obtained by adding a constant, d , to the previous term, then the sequence is called an **arithmetic sequence**, such as 2, 6, 10, 14, 18, 22,... , where $d = 4$.

d is known as the **common difference**.

Revision on Arithmetic and Geometric sequences definitions

Geometric sequence

If the rule is that each term, after the first, is obtained by multiplying the previous term by a constant, r , then the sequence is called a **geometric sequence**, such as, 4, 16, 64, 256, 1024, ...
where $r = 4$.

r is known as the **common ratio**.

Definitions

A **series** is the sum of the terms of a sequence.

A series is **finite** if it is the sum of a finite number of terms of a sequence.

A series is **infinite** if it is the sum of an infinite number of terms of a sequence.

Revision on Arithmetic and Geometric serieses definitions

Arithmetic series

An **arithmetic series** is the sum of the terms of an arithmetic sequence, such as 2, 6, 10, 14, 18, 22, . . . , with $d=4$. Denoting the first term of an arithmetic sequence with the value a and progressing by adding the value d to each previous term, the arithmetic sequence can be outlined as in Table 5.1.

Each element of a sequence can be identified by reference to its term number;

$n =$

Find the sum of the first 15 terms of the series: $20 + 18 + 16 + 14 + \dots$

$a =$

$$20 + 18 + 16 + 14 + 12 + 10 + 8 + 6 + 4 + 2 + 0 + (-2) + (-4) + (-6) + (-8)$$

$= 90$

$d = -2$

$$T_1 = a \quad , \quad d$$

$$T_2 = a + d$$

$$T_3 = a + d + d = a + 2d$$

$$T_4 = a + 2d + d = a + 3d$$

$\vdots$

$$T_n = a + (n-1)d$$



Chapter 05

Arithmetic sequence

$$T_1 + T_2 + T_3 + T_4 + \dots + T_n$$
$$\underbrace{a + (a+d) + (a+2d) + (a+3d) + \dots + (a+(n-1)d}$$

$$na + \underbrace{(1+2+3+\dots+(n-1))d}$$

$$= \frac{n(n-1)}{2}$$

Ex
 $1+2+3+4+5 = 15$

$$\frac{(5)(6)}{2} = \frac{30}{2} = 15$$

$$S_n = na + \frac{n(n-1)}{2}d$$

(5.2)

or

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

Arithmetic sequence

$$S_n = \frac{n}{2} [2a + (n - 1) d]$$

Table 5.1 Arithmetic sequence

Sequence	a	$a + d$	$a + 2d$		$a + (n - 1) d$
Term number	T_1	T_2	T_3		T_n

Find the sum of the first 15 terms of the series: $20 + 18 + 16 + 14 + \dots$

Solution:

This is an arithmetic series since the difference $d = -2$.

Therefore,

$a = 20$, $d = -2$, $n = 15$, and hence,
using formula (5.2),

$$S_{15} = \frac{15}{2} [2(20) + (15 - 1)(-2)] = 90$$

$$S_n = \frac{n}{2} [(2)(a) + (n-1)d]$$

Geometric series

Find the sum of the first $n=$ 12 terms of the series: $4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \dots$

$$r = \frac{1}{2}$$

$$a = \textcircled{4} + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512}$$

$$S_n = 7.9980$$

Geometric Series(or geometric progression denoted by GP)

$$T_1 = a$$

r = common ration

$$T_2 = ra$$

$$T_3 = r \cdot r \cdot a = r^2 a$$

$$T_4 = r \cdot r^2 a = r^3 a$$

$\vdots$

$$T_n = r^{n-1} a$$

Geometric Series(or geometric progression denoted by GP)

$$S_n = T_1 + T_2 + T_3 + T_4 + \dots + T_n$$

$$= a + ra + r^2a + r^3a + \dots + r^{(n-1)}a$$

$$rS_n = ra + r^2a + r^3a + r^4a + \dots + r^{n-1}a + r^na$$

$$r^{n-1} \cdot r$$

$$= r^{n-1+1}$$

$$= r^n$$

Geometric Series(or geometric progression denoted by GP)

$$S_n - rS_n = \underbrace{a}_{\text{yellow circle}} + \cancel{ra} + \cancel{r^2a} + \cancel{r^3a} + \dots + \cancel{r^{n-1}a} - \cancel{ra} - \cancel{r^2a} - \cancel{r^3a} - \dots - \cancel{r^{n-1}a} - \underbrace{r^na}_{\text{yellow circle}}$$

$$S_n - rS_n = a - r^na$$

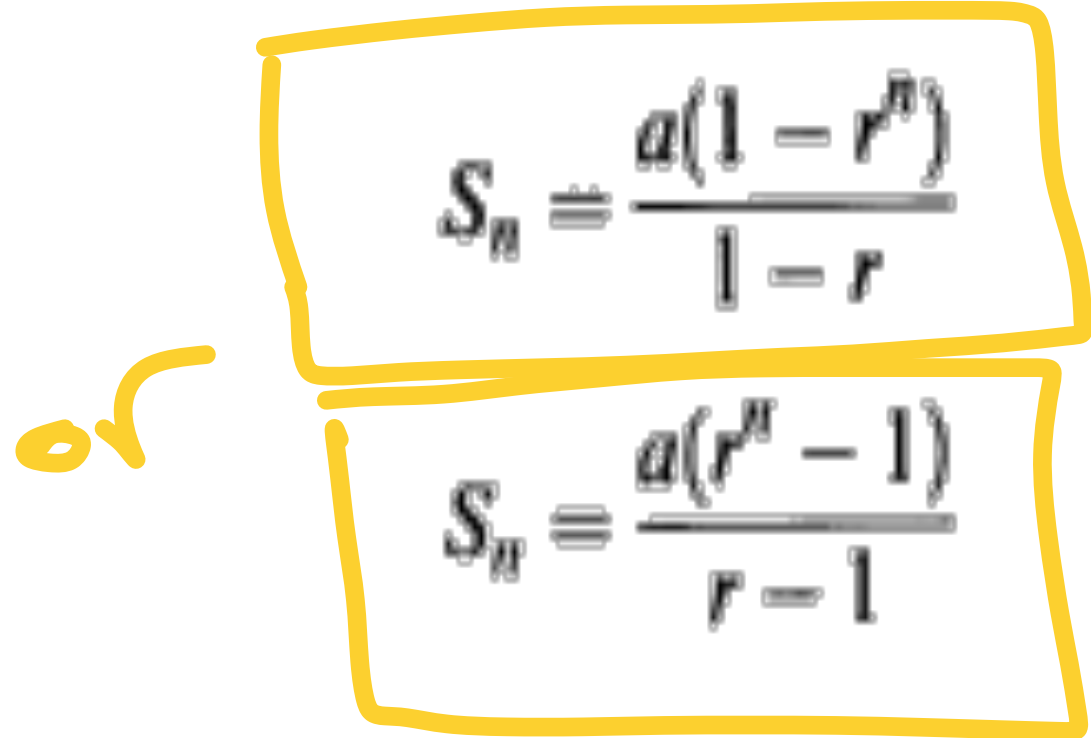
$$\cancel{S_n(1-r)} = \frac{a(1-r^n)}{1-r}$$

Chapter 05

Geometric Series(or geometric progression denoted by GP)

The sum of the first n terms of a geometric series is given by the formula,

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$


$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$r = \frac{1}{2}, \quad a = 4, \quad n = 12$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{4(1-(\frac{1}{2})^{12})}{1-\frac{1}{2}} = 7.9980$$

$q =$

A manufacturer produces 1200 computers each week. After week 1, he increases production by:

$d =$

Scheme I: 80 computers each week

Scheme II: 5% each week

$n =$

- (a) (i) Find the output in week 20 under each scheme.
- (ii) Find the total output over the first 20 weeks under each scheme.
- (b) Calculate for schemes I and II the week in which production first exceeds 8000.

Scheme I

(i) Output in week n is

$$T_n = a + (n-1)d$$

$$\begin{aligned} T_{20} &= 1200 + (20-1)(80) \\ &= 2720 \end{aligned}$$



Scheme II

(i) Output in week n is

$$T_n = ar^{n-1}$$

$$T_{20} = (1200)(1.05)^{19}$$

$$= 3039$$

$$1200 + 1260 + 1323 + \dots$$

$\uparrow$ 5% $\uparrow$ 5%
 $= 60$ $= 63$

$$a + \underbrace{ap} + a + a(1+p)p + a(1+p)$$

$$a + a \underbrace{(1+p)} + a(1+p)(1+p) + a(1+p)^2(1+p)$$

$$\vdots$$

Scheme I

(ii)

Total output in 20 weeks

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$\begin{aligned} S_{20} &= \frac{20}{2} [(2)(1200) + (20-1)(80)] \\ &= 39,200 \end{aligned}$$

Scheme II

(ii)

Total output in 20 weeks is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{20} = \frac{(1200) \left((1.05)^{20} - 1 \right)}{1.05 - 1}$$

$$= 39,677.2$$

Scheme I

(f)

The week in which production first exceeds 8000

$$T_n = a + (n-1)d$$

$$8000 = 1200 + (n-1)(80)$$

$$\frac{8000 - 1200}{80} = \frac{(n-1)(80)}{80}$$

$$\frac{6800}{80} = n-1$$

$$T_n = 8000$$

$$a = 1200$$

$$d = 80$$



$$n = \frac{6800}{80} + 1$$

$$\underline{n} = \frac{6800 + 80}{80} = \boxed{86}$$

The output for week 86 is 8,000

scheme II

(b) $T_n = ar^{n-1}$

$$\frac{8000}{1200} = \frac{(1200)(1.05)^{n-1}}{1200}$$

$$\log(6.6667) = \log(1.05)^{n-1}$$

$$\log(6.6667) = (n-1) \log(1.05)$$

$$T_n = 8000$$

$$a = 1200$$

$$r = 1.05$$

$$n = ?$$



scheme II

(b)

$$n-1 = \frac{\log(6.6667)}{\log(1.05)}$$

$$n-1 = 38.8833$$

$$n = 39.8833$$

During the week 39, out put reaches 8000 units.

$$T_n = 8000$$

$$a = 1200$$

$$r = 1.05$$

$$n = ?$$



Simple interest is a fixed percentage of the principal, P_0 , that is paid to an investor each year, irrespective of the number of years the principal has been left on deposit; that is, money invested at simple interest will increase in value by the same amount each year. So, if the investor is paid a fixed annual amount, $i\%$ of P_0 , then the amount of simple interest, I , received over t years is given by the formula

$$I = P_0 \times i \times t$$

$$1000 \quad i = 10\%$$

$$I = (1000)(0.1)(2) = 200$$

Therefore, the total value after t years, P_t is the principal plus interest and is given by

$$P_t = P_0 + (P_0 \times i \times t)$$

$$P_t = P_0 (1 + it) \quad (5.8)$$

Future
value

Present value
principle

When the total value (future value), the interest rate and time are known, the principal (present value) may be calculated by rewriting formula (5.8) as

$$P_0 (1 + it) = P_t$$

$$P_0 = \frac{P_t}{1 + it}$$

present value formula
(5.9)

Note: The interest rate is always quoted as a percentage, $i\%$. In calculations, you write $i\%$ as the fraction $i/100$

In the modern business environment, the interest on money borrowed (lent or invested) is usually compounded. Compound interest pays interest on the principal plus on any interest accumulated in previous years. The total value, P_t of a principal, P_0 , when interest is compounded at $i\%$ per annum is

$$P_t = P_0 (1+i)^t \quad (5-10)$$

$$i = 10\%$$

1000 , 1100 , 1210 , 1331 , ...

Deriving the compound interest formula

Before proceeding with worked examples on compound interest, the derivation of the compound interest formula is explained briefly:

Principal at start of year	Interest paid each year	Total at end of year = principal + interest	Total at end of year = principal at start of next year
P_0	$i \times P_0$	$P_0 + iP_0 = P_0(1 + i)$	$P_1 = P_0(1 + i)$
P_1	$i \times P_1$	$P_1 + iP_1 = [P_1] \cdot (1 + i) =$ $[P_0(1 + i)] \cdot (1 + i) = P_0(1 + i)^2$	$P_2 = P_0(1 + i)^2$
P_2	$i \times P_2$	$P_2 + iP_2 = [P_2] \cdot (1 + i) =$ $[P_0(1 + i)^2] \cdot (1 + i) = P_0(1 + i)^3$	$P_3 = P_0(1 + i)^3$
...	...	...	...
P_{t-1}	$i \times P_{t-1}$	$P_{t-1} + iP_{t-1} = [P_{t-1}] \cdot (1 + i) =$ $[P_0(1 + i)^{t-1}] \cdot (1 + i)$ $= P_0(1 + i)^t$	$P_t = P_0(1 + i)^t$



Each year, the interest earned is added to the total amount on deposit (principal, p_0 , plus any accumulated interest), at the beginning of that year.

Notice that the amounts due at the end of each year form a geometric progression where $a=p_0$, and r , the common ratio is $(1+i)$, giving the sequence

$$p_0, p_0(1+i)^1, p_0(1+i)^2, \dots, p_0(1+i)^t$$

Ex Calculate the amount owed on a loan of $\text{£}1000$ over $t=3$ years at an interest rate of 8% compounded annually.

Solution:

$$P_0 = 1000$$

$$i = \frac{8}{100} = 0.08$$

$$t = 3 \text{ years}$$

$$P_t = P_0 (1+i)^t$$

$$P_3 = (1000) (1+0.08)^3 = £1259.712$$



Present value at compound interest

The present value of a future sum, P_t , is the amount which, when put on deposit at $t = 0$, at $i\%$ rate of interest, will grow to the value of P , after t years. The present value, P_0 , is calculated by rearranging the compound interest formula (5.10):

$$P_t = P_0 (1+i)^t$$

$$P_0 = \frac{P_t}{(1+i)^t} \quad (5.11)$$

Example

£5000 is invested at 8% compound interest per annum for three years.

- (a) Calculate the value of the investment at the end of three years.
- (b) Compute the present value of receiving (i) £6298.50, (ii) £15000 in three years' time when the discount rate is 8%.

Solution:

(a) $P_0 = £5000$
 $i = 8/100 = 0.08$
 $t = 3$

By using formula (5.10).

Solution:

(a)

$$P_3 = 5000 (1 + 0.08)^3$$
$$= \text{£} 6298.5$$

Solution:

(b) The present value of receiving £ 6298.5
and £ 15000 in three years at a discount rate
of 8% is calculated by using (5.11)

solution:

(b) Present value of £6298.5

$$P_0 = \frac{P_3}{(1+i)^3} = \frac{6298.5}{(1.08)^3} = 5000$$

Solution:

(b) Present value of £15000

$$P_0 = \frac{P_3}{(1+i)^3} = \frac{15000}{(1.08)^3} = 11907.48.$$

Thank you ...