

# Mathematics

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## Lecture 5

جامعة الجميع الذكية

تسعى لتكون الكيان الأول الذي يتبنى مبدأ المجتمع التعليمي  
التعاوني ويقدم العلم مقابل العمل لتنمية وبناء المجتمعات.

MAT 112

Additional Differentiation Topics



جامعة الجميع الذكية  
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## Chapter Objectives

- To develop a differentiation formula for  $y = \ln u$ .
- To develop a differentiation formula for  $y = e^u$ .
- To give a mathematical analysis of the economic concept of elasticity.
- To discuss the notion of a function defined implicitly.
- To show how to differentiate a function of the form  $u^v$ .
- To approximate real roots of an equation by using calculus.
- To find higher-order derivatives both directly and implicitly.

- 1) Derivatives of Logarithmic Functions
- 2) Derivatives of Exponential Functions
- 3) Logarithmic Differentiation
- 4) Higher-Order Derivatives

## 1 Derivatives of Logarithmic Functions

- The derivatives of log functions are:

$$\text{a. } \frac{d}{dx}(\ln x) = \frac{1}{x} \ln \left( \lim_{h \rightarrow 0} \left( 1 + \frac{h}{x} \right)^{x/h} \right)$$

$$\text{b. } \frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ where } x \neq 0$$

$$\text{c. } \frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx} \text{ for } u \neq 0$$



**Example – Differentiating Functions Involving  $\ln x$**

*a. Differentiate  $f(x) = 5 \ln x$ .*

Solution:  $f'(x) = 5 \frac{d}{dx}(\ln x) = \frac{5}{x}$  for  $x > 0$

## Example

*b. Differentiate*  $y = \frac{\ln x}{x^2}$

Solution:

$$y' = \frac{x^2 \frac{d}{dx}(\ln x) - (\ln x) \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 \left( \frac{1}{x} \right) - (\ln x)(2x)}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3} \text{ for } x > 0$$

**Example – Rewriting Logarithmic Functions before Differentiating**

a. Find  $dy/dx$  if  $y = \ln(2x+5)^3$ .

Solution:

$$\frac{dy}{dx} = 3 \left( \frac{1}{2x+5} \right) (2)$$

$$= \frac{6}{2x+5} \text{ for } x > -5/2$$

b. Find  $f'(p)$  if  $f(p) = \ln((p+1)^2(p+2)^3(p+3)^4)$

Solution:

$$\begin{aligned} f'(p) &= 2\left(\frac{1}{p+1}\right)(1) + 3\left(\frac{1}{p+2}\right)(1) + 4\left(\frac{1}{p+3}\right)(1) \\ &= \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3} \end{aligned}$$



## Procedure to Differentiate $\log_b u$

- Convert  $\log_b u$  to  $\frac{\ln u}{\ln b}$  and then differentiate.



**Example – Differentiating a Logarithmic Function to the Base 2**

*Differentiate  $y = \log_2 x$ .*

Solution: 
$$\frac{dy}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{(\ln 2)x}$$



## 2 Derivatives of Exponential Functions

- The derivatives of exponential functions are:

a.  $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

b.  $\frac{d}{dx}(e^x) = e^x$

c.  $\frac{d}{dx}(b^u) = b^u (\ln b) \frac{du}{dx}$

d.  $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} \text{ for } f'(f^{-1}(x)) \neq 0$

e.  $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$



**Example – Differentiating Functions Involving  $e^x$**

a. Find  $\frac{d}{dx}(3e^x)$ .

Solution:  $\frac{d}{dx}(3e^x) = 3 \frac{d}{dx}(e^x) = 3e^x$



b. If  $y = \frac{x}{e^x}$ , find  $\frac{dy}{dx}$ .

Solution:  $\frac{dy}{dx} = e^{-x} \frac{d}{dx} x + x \frac{d}{dx} (e^{-x}) = \frac{1-x}{e^x}$



c. Find  $y'$  when  $y = e^2 + e^x + \ln 3$

Solution:  $y' = 0 + e^x + 0 = e^x$

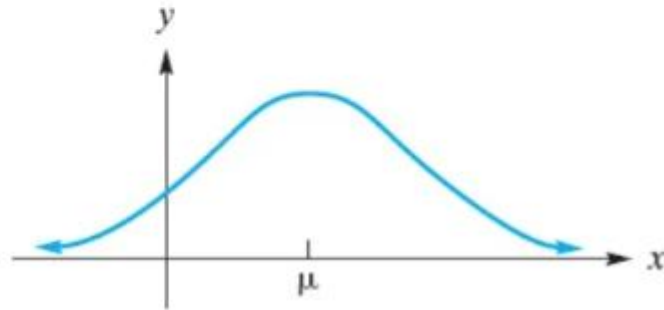


**Example – The Normal-Distribution Density Function**

*Determine the rate of change of  $y$  with respect to  $x$*

*when  $x = \mu + \sigma$ .*

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



**Example – Differentiating Different Forms**

Find  $\frac{d}{dx}(e^2 + x^e + 2^{\sqrt{x}})$

Solution:

$$\begin{aligned}\frac{d}{dx}(e^2 + x^e + 2^{\sqrt{x}}) &= ex^{e-1} + [e^{(\ln 2)\sqrt{x}}](\ln 2)\left(\frac{1}{2\sqrt{x}}\right) \\ &= ex^{e-1} + \frac{2^{\sqrt{x}} \ln 2}{2\sqrt{x}}\end{aligned}$$

## Higher-Order Derivatives

For higher-order derivatives:

|                    |           |              |                      |                          |           |
|--------------------|-----------|--------------|----------------------|--------------------------|-----------|
| First derivative:  | $y'$      | $f'(x)$      | $\frac{dy}{dx}$      | $\frac{d}{dx}(f(x))$     | $D_x y$   |
| Second derivative: | $y''$     | $f''(x)$     | $\frac{d^2 y}{dx^2}$ | $\frac{d^2}{dx^2}(f(x))$ | $D_x^2 y$ |
| Third derivative:  | $y'''$    | $f'''(x)$    | $\frac{d^3 y}{dx^3}$ | $\frac{d^3}{dx^3}(f(x))$ | $D_x^3 y$ |
| Fourth derivative: | $y^{(4)}$ | $f^{(4)}(x)$ | $\frac{d^4 y}{dx^4}$ | $\frac{d^4}{dx^4}(f(x))$ | $D_x^4 y$ |

**Example 1 – Finding Higher-Order Derivatives**

*a. If  $f(x) = 6x^3 - 12x^2 + 6x - 2$ , find all higher-order derivatives.*

Solution:

$$f'(x) = 18x^2 - 24x + 6$$

$$f''(x) = 36x - 24$$

$$f'''(x) = 36$$

$$f^{(4)}(x) = 0$$



*b. If  $f(x) = 7$ , find  $f'(x)$ .*

Solution:

$$f'(x) = 0$$

$$f''(x) = 0$$



$$f(x) = 3 \ln(x + 2) + 4 \ln(x + 1) + 2 \ln(x - 2)$$

$$f'(x) = \frac{3}{x+2} + \frac{4}{x+1} + \frac{2}{x-2}$$



$$f(x) = 2x^2 \ln x$$

$$f'(x) = (4x) (\ln x) + \left(\frac{1}{x}\right) 2x^2$$



**Differentiate the following functions:**

$$f(x) = e^{ax^2}$$

$$f'(x) = e^{ax^2} (2ax)$$



**Differentiate the following functions:**

$$f(x) = e^{x^2-2x+1}$$

$$f'(x) = e^{x^2-2x+1}[2x-2] = (2x-2)e^{x^2-2x+1}$$

$$f(x) = \ln x^2 = 2 \ln x$$

$$f'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$$



$$y = -x - x^2$$

$$\frac{dy}{dx} = -1 - 2x$$



$$y = x^3 + e^x$$

$$y' = 3x^2 + e^x$$



$$y = 8 - x$$

$$\frac{dy}{dx} = -1$$



$$f(x) = 3x^2 + 2x$$

$$f'(x) = 6x + 2$$



$$f(x) = \frac{1}{\sqrt{x}}$$

$$f'(x) = \frac{-1}{2} x^{\frac{-3}{2}}$$



$$f(x) = \left(\frac{6}{7}\right)^{\frac{2}{3}}$$

$$f'(x) = 0$$



$$f(x) = \frac{3}{5}x^6$$

$$f'(x) = \frac{18}{5}x^5$$



$$y = \frac{x+1}{x-1} \rightarrow y''$$

$$y' = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$
$$= -\frac{2}{(x-1)^2} = -2(x-1)^{-2}$$

$$y'' = 4(x-1)^{-3}$$
$$= \frac{4}{(x-1)^3}$$



$$f(x) = \sqrt{x} \rightarrow f''(x)$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{\frac{3}{2}}}$$



**Find  $f'''(x)$  :**

If  $f(x) = 3x^2 + 2x$

**Solution**

$$f'(x) = 6x + 2$$

$$f''(x) = 6$$

$$f'''(x) = \text{zero}$$



$$y = x^5 + x^4 + x^3 + x^2 + x + 1 \rightarrow y'''$$

$$y' = 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

$$y'' = 20x^3 + 12x^2 + 6x + 2$$

$$y''' = 60x^2 + 24x + 6$$



$$y = 4x^3 - 12x^2 + 6x + 2 \rightarrow y'''$$

$$y' = 12x^2 - 24x + 6$$

$$y'' = 24x - 24$$

$$y''' = 24$$



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