

Mathematics 3 – MAT132

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Lecture 4 – Introduction to Fourier Series



Lecture Objective

- Fourier Series Formula
- Derivatives of Basic Trigonometric Functions
- Derivatives for other functions with Trigonometric Functions
- Important Relations for Trigonometric Functions
- Extra Practice

Fourier Series Formula

First let us write down a full series of sines and cosines, with a name for all coefficients:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx \frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx \frac{\pi}{L}\right)$$

Where:

- $f(x)$ is the function we want (such as a square wave)
- L is **half of the period** of the function
- a_0 , a_n and b_n are **coefficients** that we need to calculate!

Important knowledge for Fourier Series

Derivatives of Basic Trigonometric Functions

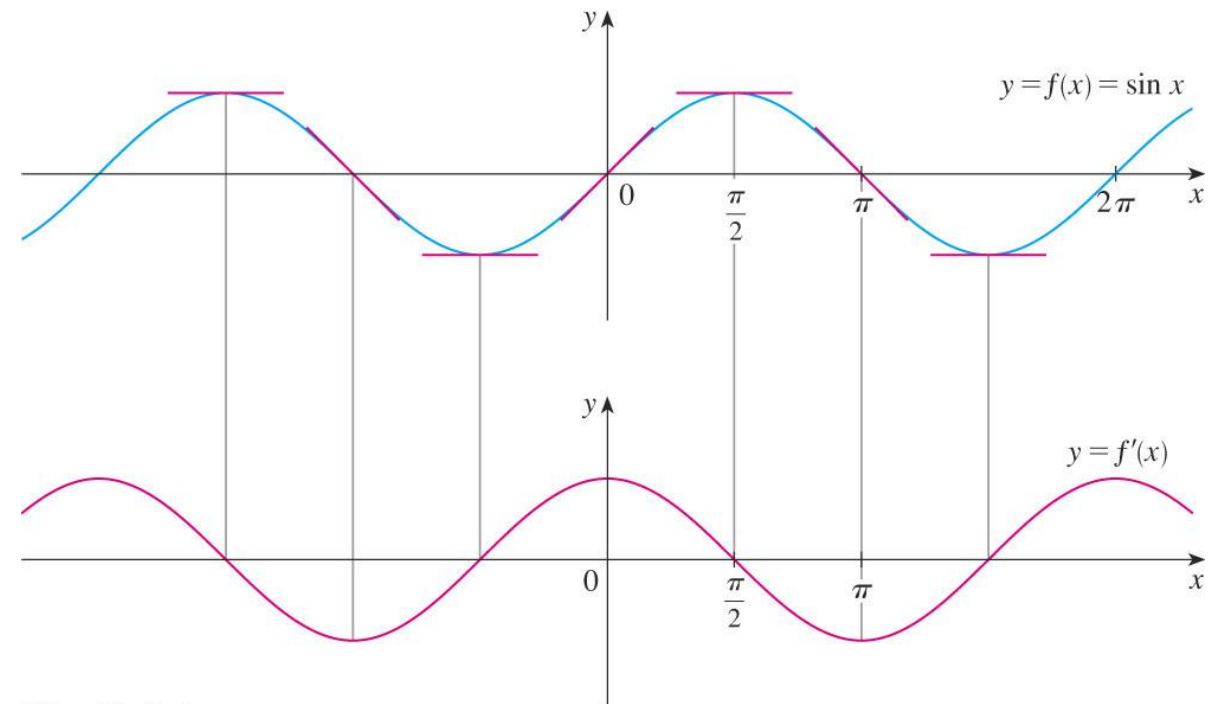
Derivatives of Basic Trigonometric Functions

Let's sketch the graph of the function:

$f(x) = \sin x$ and use the interpretation of $f'(x)$ as the slope of the tangent to the sine curve in order to sketch the graph of f' .

Derivatives of Basic Trigonometric Functions

Then, it looks as if the graph of f' may be the same as the cosine curve.



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Rules for Derivatives of Basic Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of Basic Trigonometric Functions

Let's try to confirm
our guess that, if $f(x) = \sin x$,
then $f'(x) = \cos x$.

Derivatives of Basic Trigonometric Functions

From the definition of a derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right] \\ &= \lim_{h \rightarrow 0} \left[\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right] \\ &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \end{aligned}$$

Derivatives of Basic Trigonometric Functions

$$\lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h}$$

Two of these four limits are easy to evaluate.

Derivatives of Basic Trigonometric Functions

Since we regard x as a constant when computing a limit as $h \rightarrow 0$, we have:

$$\lim_{h \rightarrow 0} \sin x = \sin x$$

$$\lim_{h \rightarrow 0} \cos x = \cos x$$

Derivatives of Basic Trigonometric Functions

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \sin x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \lim_{h \rightarrow 0} \cos x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\&= (\sin x) \cdot 0 + (\cos x) \cdot 1 \\&= \cos x\end{aligned}$$

Important relations I

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

$$\tan \theta \cot \theta = 1$$

Important relations II

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\csc^2 \theta - \cot^2 \theta = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$$

$$1 + \cot^2 \theta = \frac{1}{\sin^2 \theta}$$

Extra Practice

Extra Practice – Example 1

Differentiate:

$$y = x^2 \sin x.$$

Example 1 Solution

$$y = x^2 \sin x.$$

Using the Product Rule and Formula 4, we have:

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2) \\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

Extra Practice – Example 2

Differentiate

$$y = \tan x.$$

Example 2 Solution

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\&= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \\&\frac{d}{dx}(\tan x) = \sec^2 x\end{aligned}$$

Extra Practice – Example 3

Differentiate

$$f(x) = \frac{\sec x}{1 + \tan x}$$

Example 3 Solution

$$\begin{aligned} f'(x) &= \frac{(1 + \tan x) \frac{d}{dx} (\sec x) - \sec x \frac{d}{dx} (1 + \tan x)}{(1 + \tan x)^2} \\ &= \frac{(1 + \tan x) \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2} \\ &= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2} \end{aligned}$$

Extra Practice – Example 4

Find:

The 27th derivative of $\cos x$.

Example 4 Solution

The first few derivatives of $f(x) = \cos x$ are as follows:

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f^{(4)}(x) = \cos x$$

$$f^{(5)}(x) = -\sin x$$

Remark for Example 4

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4.

Therefore, $f^{(24)}(x) = \cos x$

Differentiating three more times, we have:

$$f^{(27)}(x) = \sin x$$

Extra Practice – Example 5

Find y'

$$y = x^3 \cot x.$$

Example 5 Solution

$$y = x^3 \cot x.$$

$$y' = 3x^2 \cot x + x^3 (-\csc^2 x)$$

$$= 3x^2 \cot x - x^3 \csc^2 x$$

Extra Practice – Example 6

Find $f'\left(\frac{\pi}{4}\right)$

if $f(x) = 5 \sin x + \cos x$.

Example 6 Solution

$$f(x) = 5 \sin x + \cos x.$$

$$f'(x) = 5 \cos x - \sin x$$

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= 5 \cos \frac{\pi}{4} - \sin \frac{\pi}{4} \\ &= \frac{5\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \frac{4\sqrt{2}}{2} \\ &= 2\sqrt{2} \end{aligned}$$

Extra Practice – Example 7

Differentiate

$$g(x) = 3 \sec(x) - 10 \cot(x)$$

Example 7 Solution

$$g(x) = 3 \sec(x) - 10 \cot(x)$$

$$\begin{aligned} g'(x) &= 3 \sec(x) \tan(x) - 10 (-\csc^2(x)) \\ &= 3 \sec(x) \tan(x) + 10 \csc^2(x) \end{aligned}$$

Extra Practice – Example 8

Differentiate

$$h(w) = 3w^{-4} - w^2 \tan(w)$$

Example 8 Solution

$$h(w) = 3w^{-4} - w^2 \tan(w)$$

$$\begin{aligned} h'(w) &= -12w^{-5} - (2w \tan(w) + w^2 \sec^2(w)) \\ &= -12w^{-5} - 2w \tan(w) - w^2 \sec^2(w) \end{aligned}$$

Extra Practice – Example 9

Differentiate

$$y = 5 \sin(x) \cos(x) + 4 \csc(x)$$

Example 9 Solution

$$y = 5 \sin(x) \cos(x) + 4 \csc(x)$$

$$y' = 5 \cos(x) \cos(x) + 5 \sin(x) (-\sin(x)) - 4 \csc(x) \cot(x)$$

$$= 5\cos^2(x) - 5\sin^2(x) - 4 \csc(x) \cot(x)$$

Extra Practice – Example 10

Differentiate

$$P(t) = \frac{\sin(t)}{3 - 2 \cos(t)}$$

Example 10 Solution

$$\begin{aligned}P'(t) &= \frac{\cos(t)(3 - 2\cos(t)) - \sin(t)(2\sin(t))}{(3 - 2\cos(t))^2} \\&= \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{(3 - 2\cos(t))^2} \\&= \frac{3\cos(t) - 2(\cos^2(t) + \sin^2(t))}{(3 - 2\cos(t))^2} \\&= \frac{3\cos(t) - 2}{(3 - 2\cos(t))^2}\end{aligned}$$

Extra Practice – Example 11

Find the derivative of

$$f(x) = 5x^3 \sin x$$

Example 11 Solution

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

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