

Business Mathematics

Prepared by Dr. Faisal Hasan

Lecture NO 10

جامعة الجميع الذكية

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MTH 112

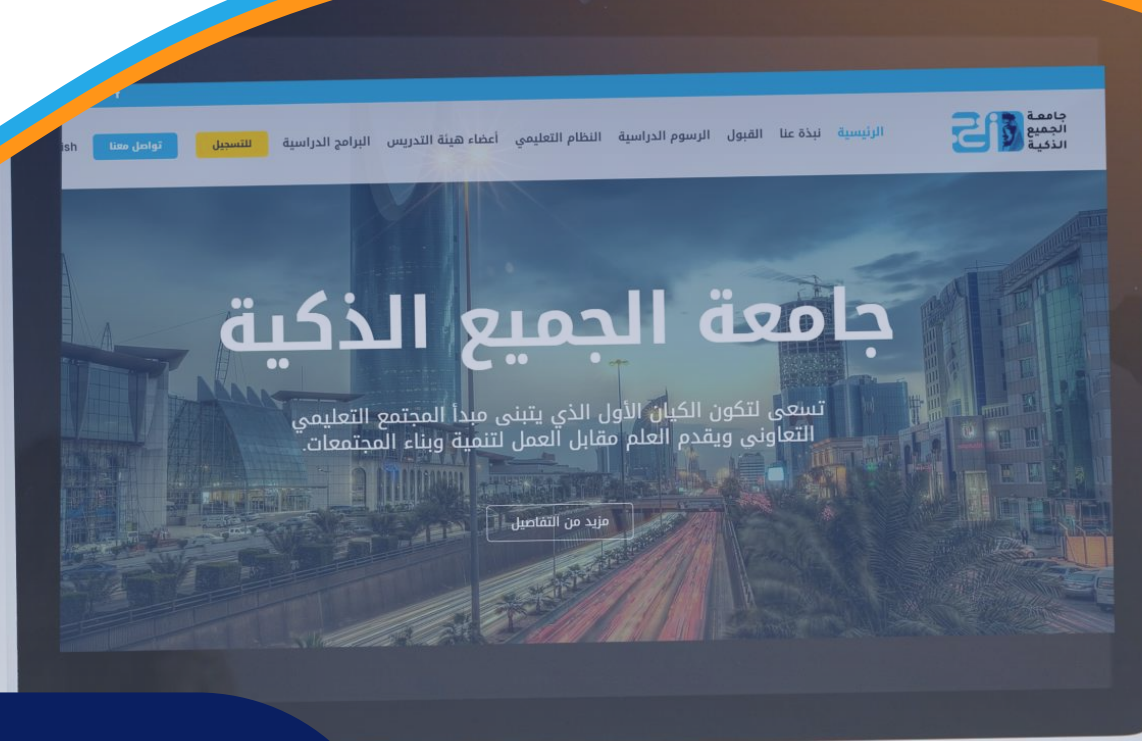
Business Mathematics



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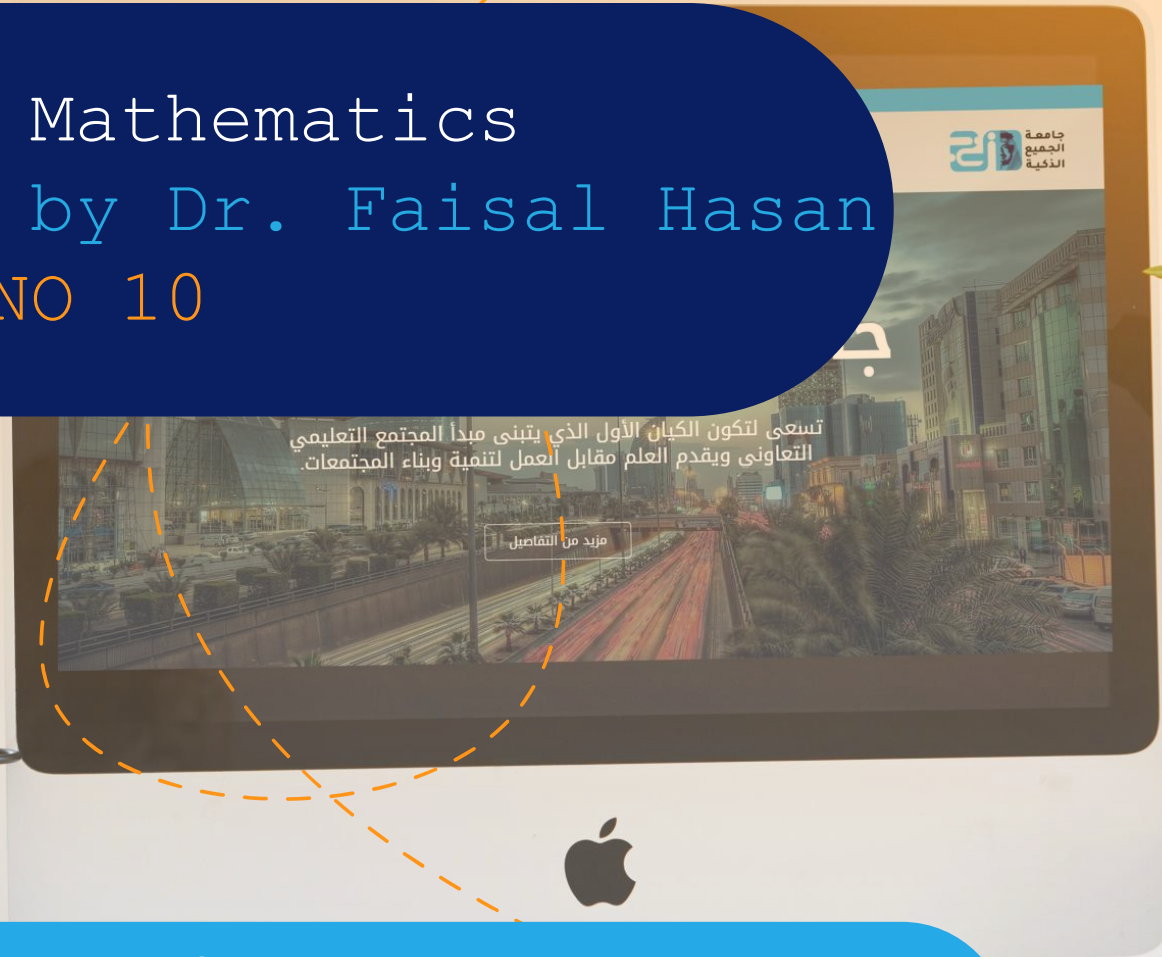
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Business Mathematics

Topics

- Chapter 1: Mathematical preliminaries
- Chapter 2: The straight line and applications
- Chapter 3: Functions and Graphs
- Chapter 4: Limits and Continuity
- Chapter 5: Financial Mathematics
- Chapter 6: Differentiation
- Chapter 7: Methods of Differentiation
- Chapter 8: Function of several variables
- Chapter 9: Integration and applications
- Chapter 10: Linear algebra and applications

Matrices Notation and Terminology

- A general $m \times n$ matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} .

- A matrix A with n rows and n columns is called a **square matrix of order n** , and the shaded entries $a_{11}, a_{22}, \dots, a_{nn}$ are said to be on the **main diagonal** of A .

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size,
then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j .

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

- If $x=5$, then $A=B$.
- For all other values of x , the matrices A and B are not equal.
- There is no value of x for which $A=C$ since A and C have different sizes.

- Rectangular display of vectors in rows and columns
- Can inform about the same vector intensity at different times or different voxels at the same time
- Vector is just a $n \times 1$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 8 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

Square (3 x 3)

Rectangular (3 x 2)

d_{ij} : i^{th} row, j^{th} column

Defined as rows x columns (R x C)

If A is any matrix and c is any scalar, then the **product** cA is the matrix obtained by multiplying each entry of the matrix A by c . The matrix cA is said to be the **scalar multiple** of A .

In matrix notation, if $A = [a_{ij}]$, then

$$(cA)_{ij} = ca_{ij}$$

Chapter 10

Linear Algebra and Applications

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix}$$

$$2\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 10 & 8 & 2 \\ 12 & 14 & 8 \end{bmatrix}$$



Chapter 10

Linear Algebra and Applications

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ 0 & 7 \\ 3 & 8 \end{bmatrix}$$

$$-3\mathbf{C} = \begin{bmatrix} -3 & -12 \\ 0 & -21 \\ -9 & -24 \end{bmatrix}$$

- For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

- We have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

- It common practice to denote $(-1)B$ by $-B$.

- If A and B are matrices of the same size, then the **sum** $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A .
- Vice versa, the **difference** $A-B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A .
- Note: Matrices of different sizes cannot be added or subtracted.

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$$

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}, \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

- The expressions $A+C$, $B+C$, $A-C$, and $B-C$ are undefined.

Addition

- Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+0 \\ 2+3 & 5+1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Subtraction

- By adding a negative matrix

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Transposition

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{b}^T = [1 \quad 1 \quad 2] \quad \mathbf{d} = [3 \quad 4 \quad 9] \quad \mathbf{d}^T = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}$$

column $\longrightarrow$ row row $\longrightarrow$ column

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 7 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

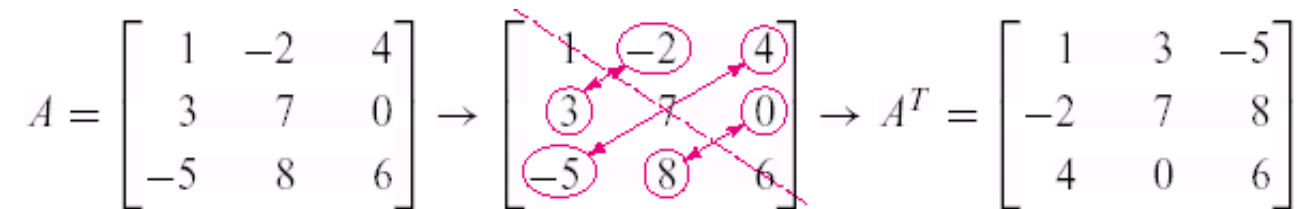
$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix} \quad \mathbf{A}^T = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 7 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\left(\mathbf{A}^T\right)^T = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix} = \mathbf{A}$$

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ 3 \ 5], \quad D = [4]$$
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

- Observe that
- In the special case where A is a **square** matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$


Definition

- If A is a square matrix, then the **trace of A** , denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

- Multiplication method:

Sum over product of respective rows and columns

$$\begin{array}{ccc}
 \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \\ \mathbf{A} \end{array} & \times & \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \\ \mathbf{B} \end{array} \\
 & & = \begin{array}{c} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \\ \text{Define output} \\ \text{matrix} \end{array} \\
 & & = \begin{array}{c} \begin{bmatrix} (1 \times 2) + (0 \times 3) & (1 \times 1) + (0 \times 1) \\ (2 \times 2) + (3 \times 3) & (2 \times 1) + (3 \times 1) \end{bmatrix} \end{array} \\
 & & = \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 13 & 5 \end{bmatrix} \end{array}
 \end{array}$$

- Matrix multiplication is NOT commutative
- $AB \neq BA$
- Matrix multiplication IS associative
- $A(BC) = (AB)C$
- Matrix multiplication IS distributive
- $A(B+C) = AB+AC$
- $(A+B)C = AC+BC$

Identity matrix

Is there a matrix which plays a similar role as the number 1 in number multiplication?

Consider the $n \times n$ matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix inverse

- **Definition.** A matrix A is called **nonsingular** or **invertible** if there exists a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I_2$$

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix inverse $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = (ad - bc) \neq 0$$

Singular matrix if $\det(A) = 0$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\det(A) = (3)(2) - (1)(5) = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix}$$

$$\det(A) = (4)(3) - (-3)(2) = 12 + 6 = 18$$

$$A^{-1} = \frac{1}{18} \begin{bmatrix} 3 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{18} & \frac{3}{18} \\ \frac{-2}{18} & \frac{4}{18} \end{bmatrix}$$

Find the inverse of the matrix

$$B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

$$\det(B) = (4)(2) - (1)(8) = 0$$

$\det(B) = 0$, Matrix B is Singular matrix

Matrix B has no inverse

Determinants

If $A = [a_{ij}]$ is a square matrix of order 1,
then $|A| = |a_{11}| = a_{11}$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example

Evaluate the determinant : $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$

Evaluate the determinant: $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$

$$\text{Solution : } \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = 4 \times 5 - 2 \times (-3) = 20 + 6 = 26$$

Solution of System of 2 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1x + b_1y = c_1 \quad \dots(i)$$

$$a_2x + b_2y = c_2 \quad \dots(ii)$$

Then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ provided $D \neq 0$,

$$\text{where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Chapter 10

Linear Algebra and Applications

$$a_1x + b_1y = c_1 \quad \dots(i)$$

$$a_2x + b_2y = c_2 \quad \dots(ii)$$

$$\text{where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Note :

If $D \neq 0$,

then the system is consistent and has unique solution.

Example

Using Cramer's rule , solve the following
system of equations $2x-3y=7$, $3x+y=5$



Using Cramer's rule , solve the following
system of equations $2x-3y=7$, $3x+y=5$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 2 + 9 = 11 \neq 0$$

$$D_1 = \begin{vmatrix} 7 & -3 \\ 5 & 1 \end{vmatrix} = 7 + 15 = 22$$

$$D_2 = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = 10 - 21 = -11$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{22}{11} = 2 \text{ and } y = \frac{D_2}{D} = \frac{-11}{11} = -1$$

Example

Using Cramer's rule , solve the following
system of equations $x+2y=10$, $5x+8y=40$



Using Cramer's rule , solve the following
system of equations $x+2y=10$, $5x+8y=40$

$$D = \begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix} = 8 - 10 = -2 \neq 0$$

$$D_1 = \begin{vmatrix} 10 & 2 \\ 40 & 8 \end{vmatrix} = 80 - 80 = 0$$

$$D_2 = \begin{vmatrix} 1 & 10 \\ 5 & 40 \end{vmatrix} = 40 - 50 = -10$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{0}{-2} = 0 \text{ and } y = \frac{D_2}{D} = \frac{-10}{-2} = 5$$

Example

Using Cramer's rule , solve the following
system of equations $3x+y=12$, $5x-y=4$



Using Cramer's rule , solve the following
system of equations $3x+y=12$, $5x-y=4$

$$D = \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = -3 - 5 = -8 \neq 0$$

$$D_1 = \begin{vmatrix} 12 & 1 \\ 4 & -1 \end{vmatrix} = -12 - 4 = -16$$

$$D_2 = \begin{vmatrix} 3 & 12 \\ 5 & 4 \end{vmatrix} = 12 - 60 = -48$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{-16}{-8} = 2 \text{ and } y = \frac{D_2}{D} = \frac{-48}{-8} = 6$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

Find $3\mathbf{A}$ and $\text{Tr}(\mathbf{A})$ and $\mathbf{A}^T$

Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 5 & 2 \end{bmatrix}$$



Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Find $A + B$, $A - B$ and AB

Using Cramer's rule , solve the following
system of equations $2x+y=1$, $4x-y=5$