



مجموعة جامعة
الجميع الذكية
للتعليم القابضة

MIDOCEAN
UNIVERSITY
BY EVERYONE'S SMART UNIVERSITY

Mathematical Foundations For Data Science and AI

Lecture 07

AI102

Linear Algebra; Matrices

By
Dr. Khaled ELMenshawy

www.midocean.ae

LEARNING OUTCOMES

- Differentiate Scalar, Vector, and Matrices.
- Understanding The Norms.
- Understanding Distances between Vectors.
- Understanding sum, differences, and multiplying of the Vector.
- Understanding The Dot Product.
- Understanding Addition and Subtraction of Matrices
- Understanding The Transpose of a Matrix.
- Understanding The Identity Matrix.
- Understanding The Determinant of the Matrix.
- Understanding The Inverse of a Matrix
- Understanding a linear Transformation .
- Apply Neural networks using the Matrices.

***The Matrix** is a collection of numbers ordered in rows and columns. We would normally denote a matrix with a capital letter, for instance, A*

$$A_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

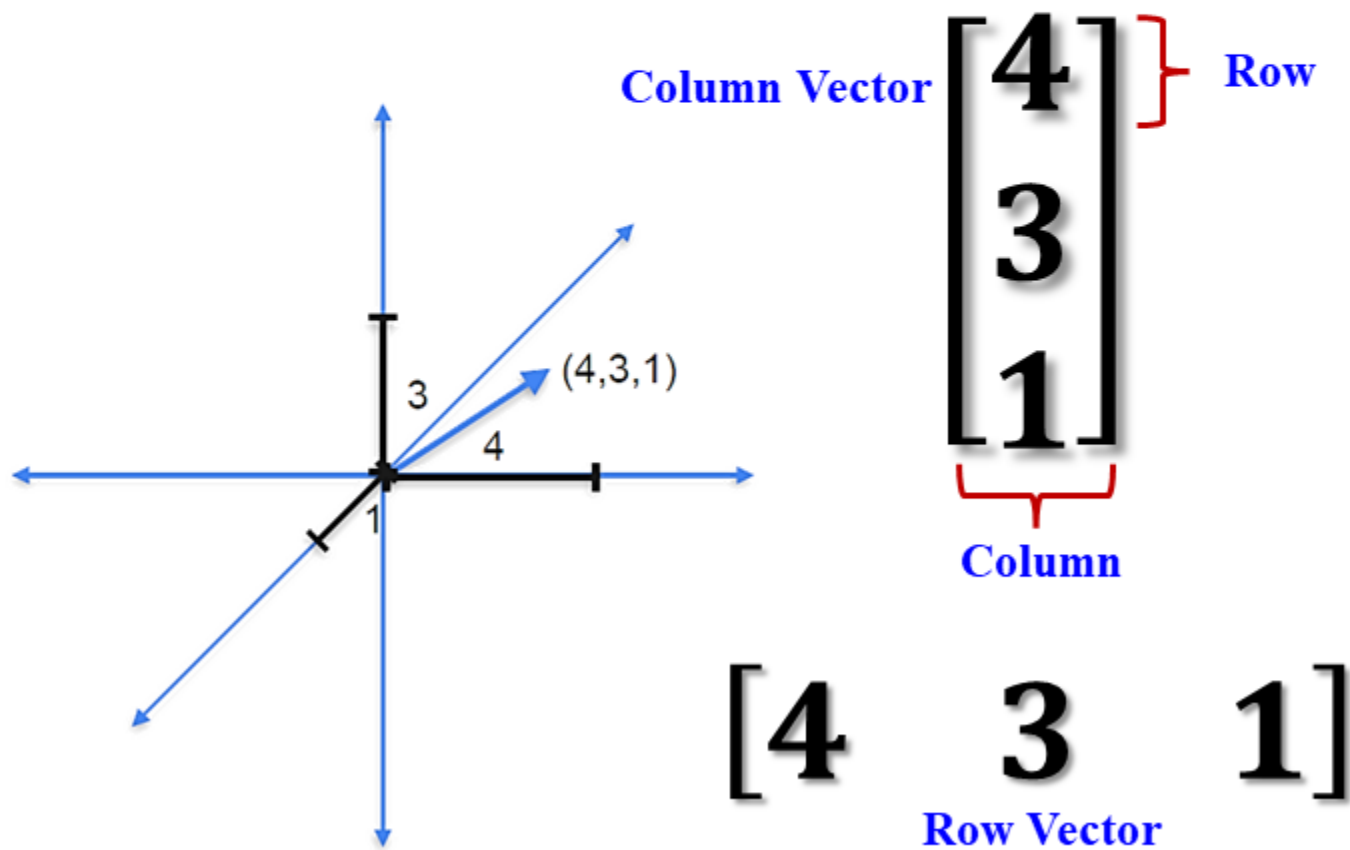
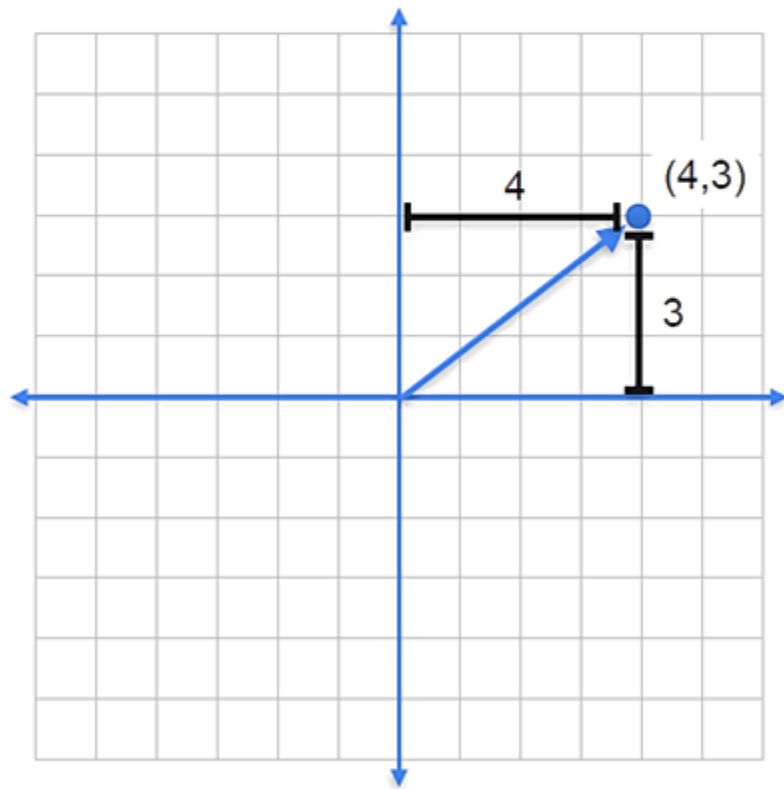
3 Rows

3 Columns

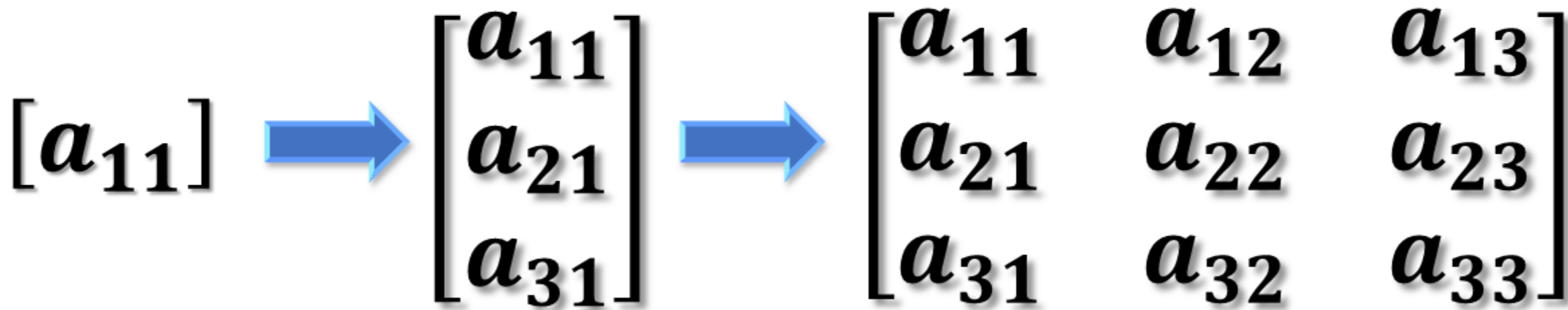
***The Scalar** is a matrix of one row and one column, and it has no direction. all numbers we know from algebra are referred to as scalars in linear algebra.*

$$\underbrace{[a_{11}]}_{\text{Column}} \Bigg\} \text{Row}$$

*The **Vector** is a simpler way of number that only has one column. It turns out that vectors can be seen as arrows in the plane or in a higher dimensional space. Two very important components of vectors are their magnitude, and their direction.*



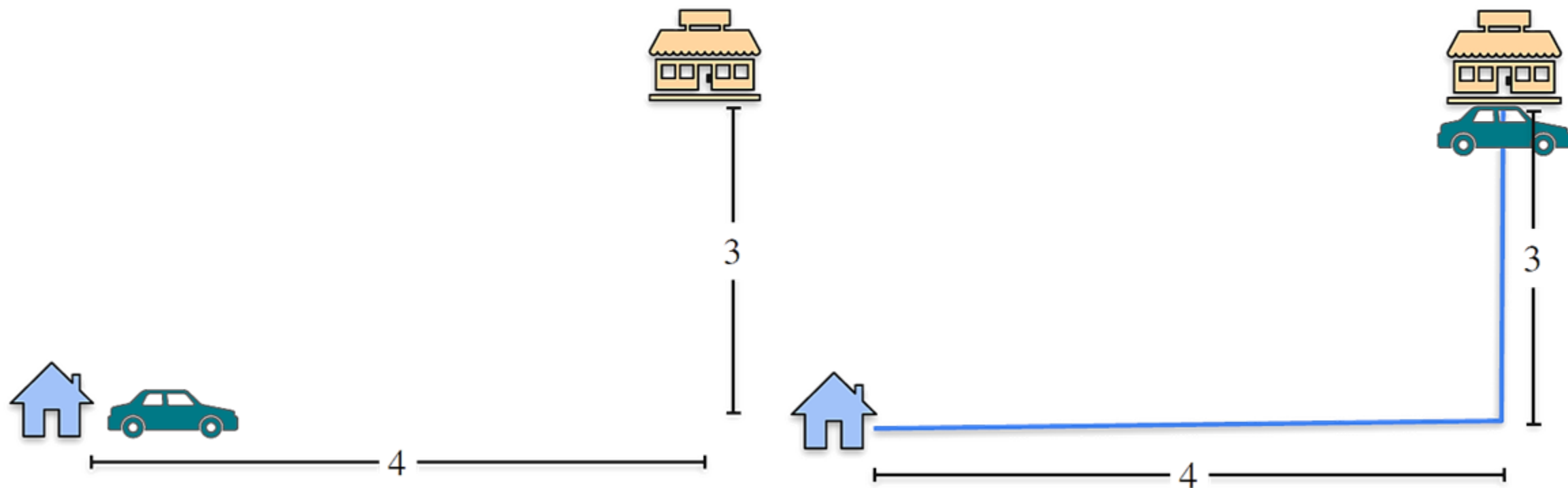
The Vector is a group of scalars, and *the matrix* is a group of Vectors.

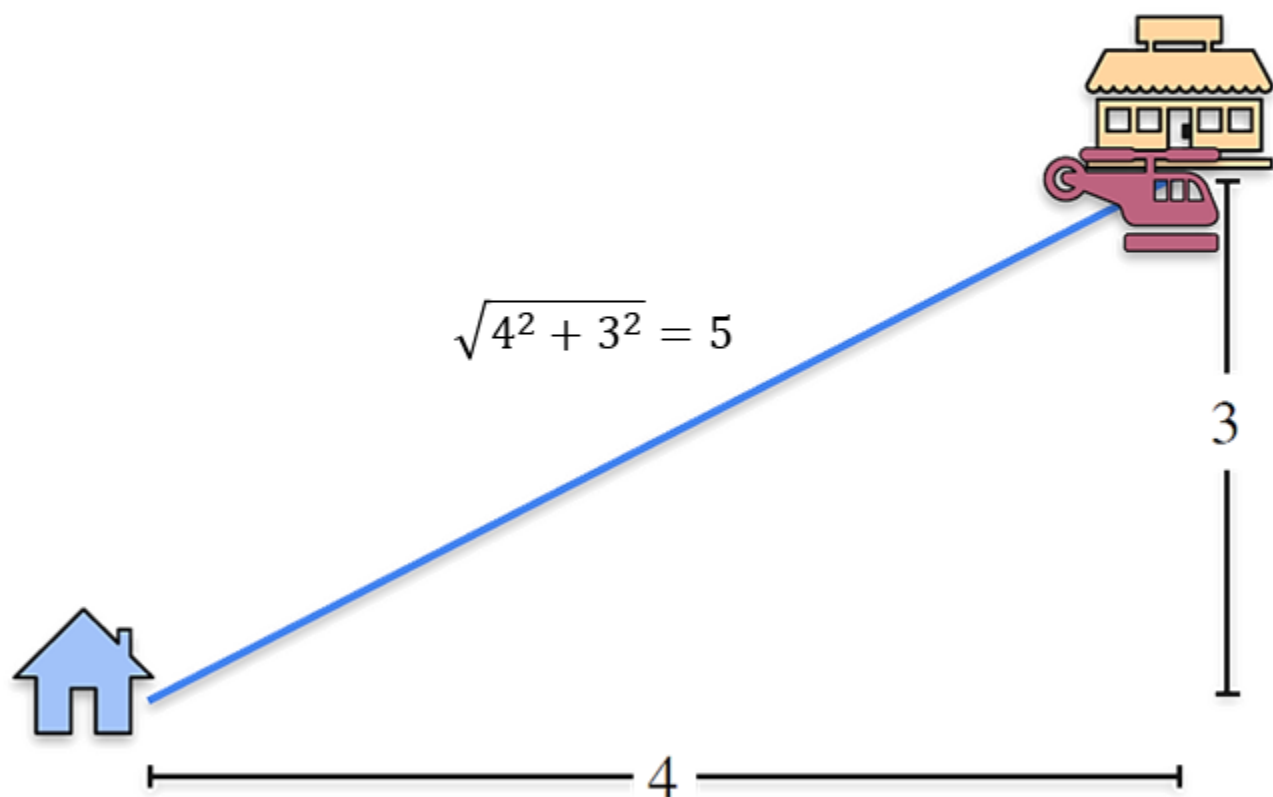


The Scalar

The Vector

The matrix



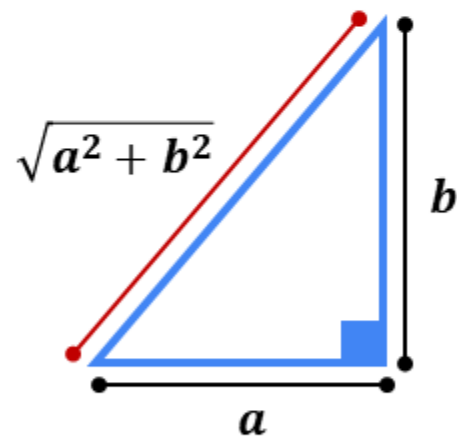


Pythagorean theorem

The length of the hypotenuse in a right triangle is the square root of the sum of the squares off both sides.

$$\text{hypotenuse}(c^2) = a^2 + b^2$$

$$\therefore c = \sqrt{a^2 + b^2}$$



***A Norm** is a way to measure the size of a vector, a matrix, or a tensor.*

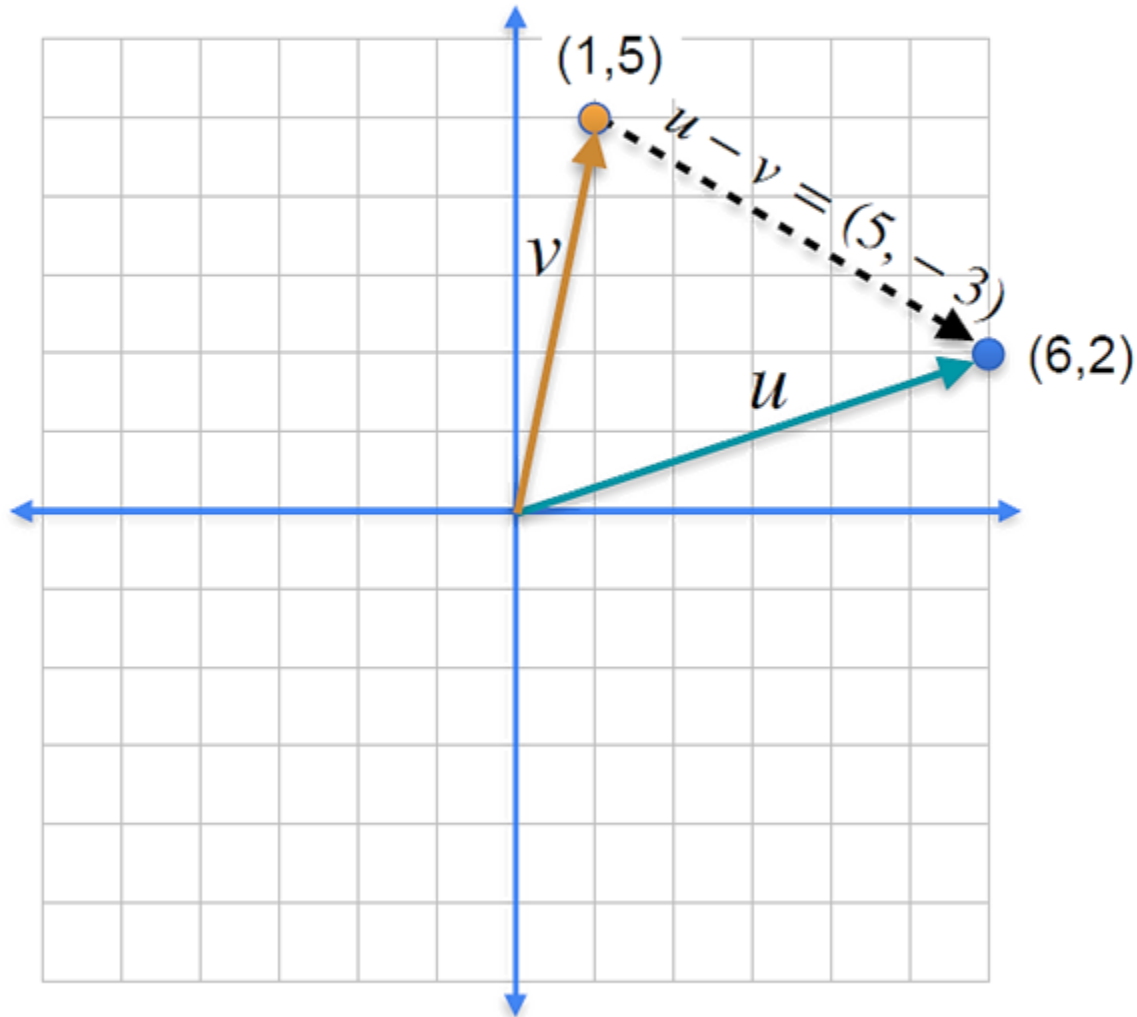


$$L1 - Norm = |(a, b)| = |a| + |b|$$



$$L2 - Norm = |(a, b)| = \sqrt{a^2 + b^2}$$

Distances between Vectors



L1-distance

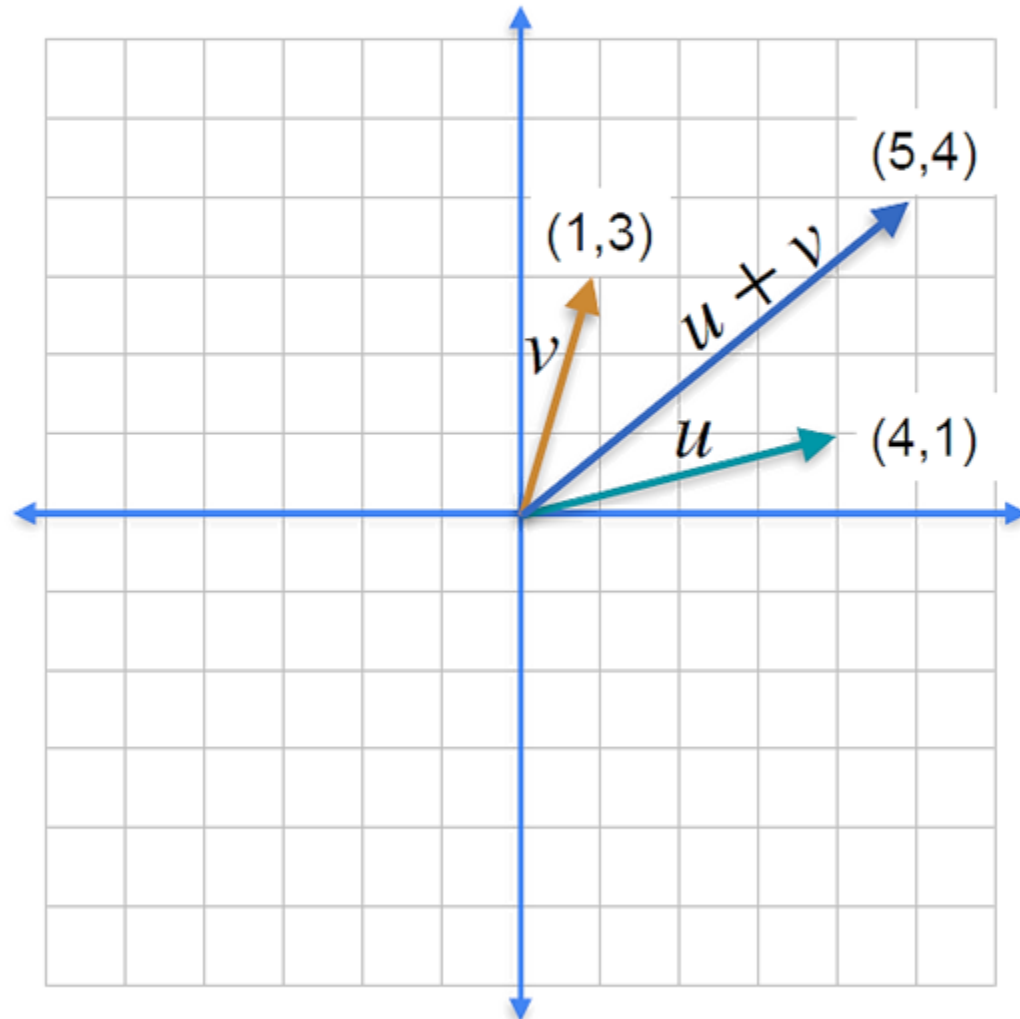
$$L1 - Norm = |u - v| = |5| + |-3| = 8$$



L2-distance

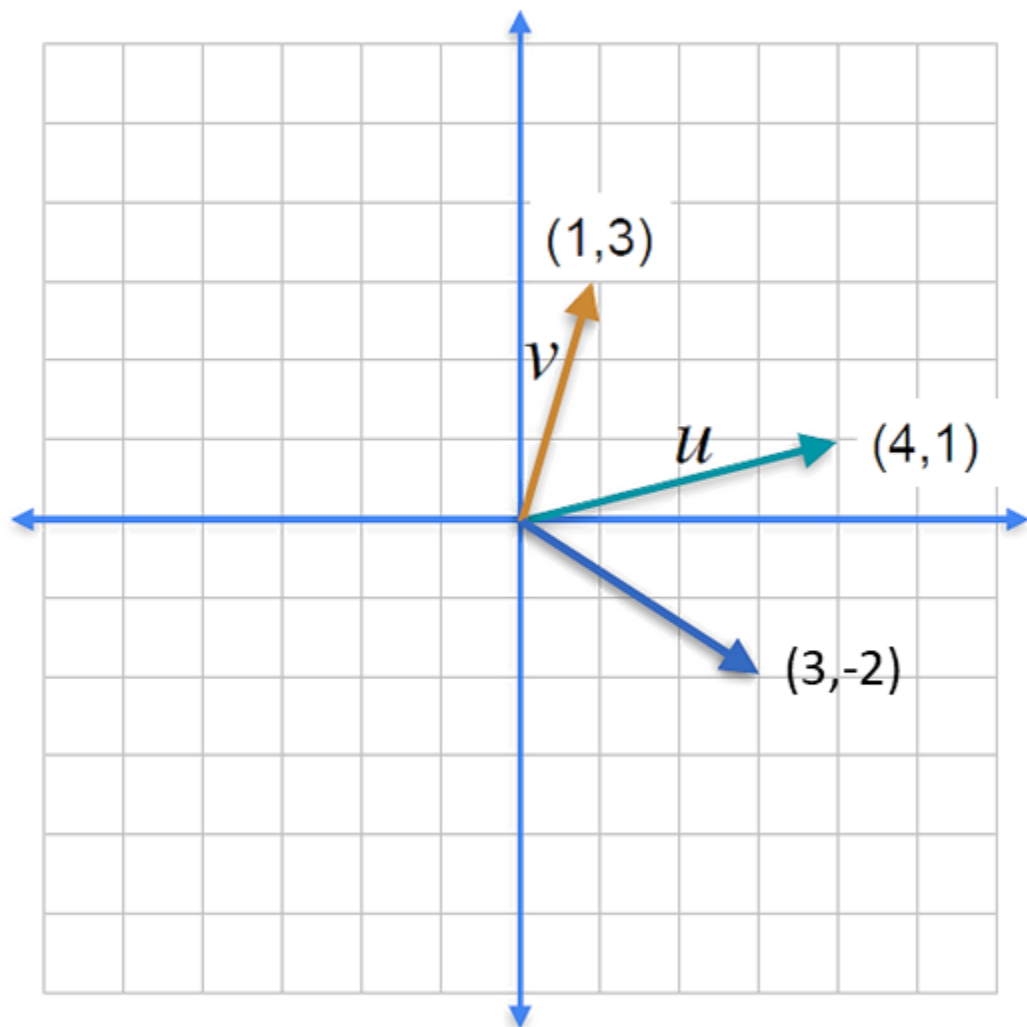
$$L2 - Norm = |u - v| = \sqrt{5^2 + (-3)^2} = 5.83$$

Sum of Vectors



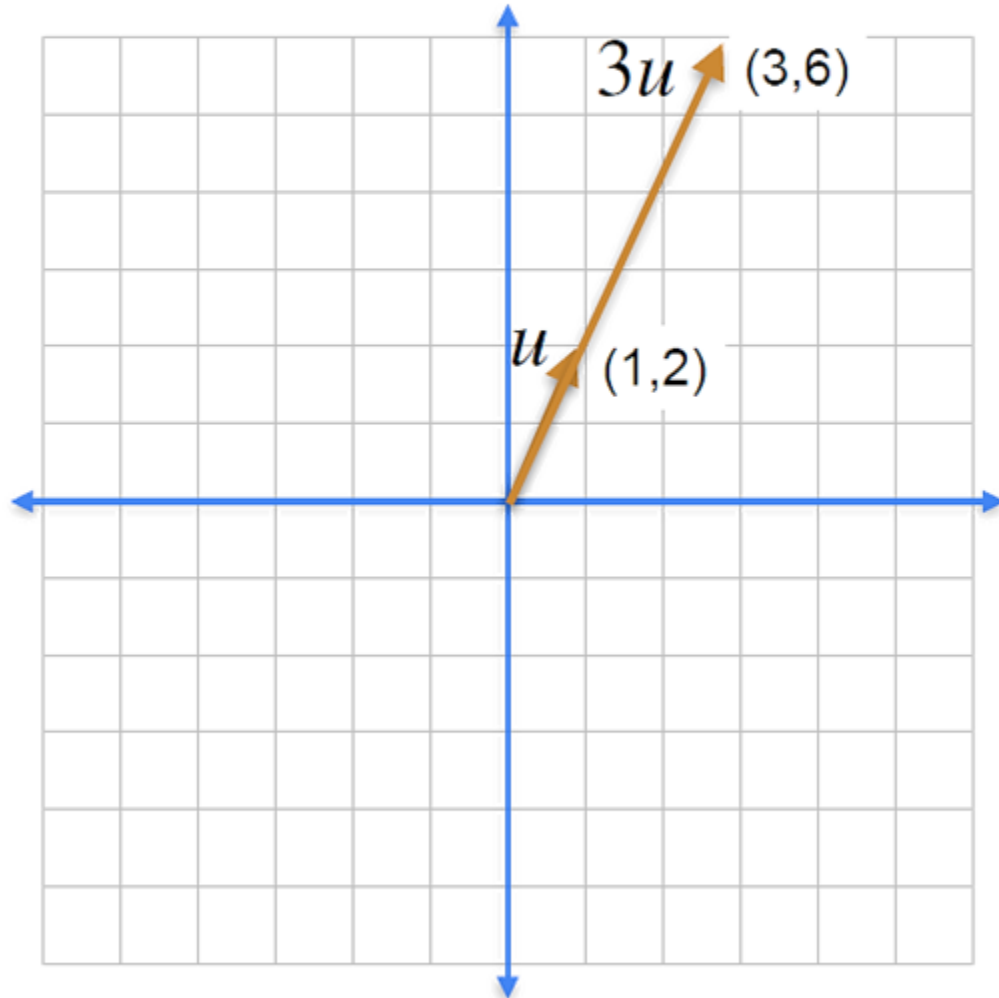
$$u + v = (4 + 1), (1 + 3) = (5, 4)$$

Difference of Vectors



$$u - v = (4 - 1), (1 - 3) = (3, -2)$$

Multiplying a Vector by a Scalar

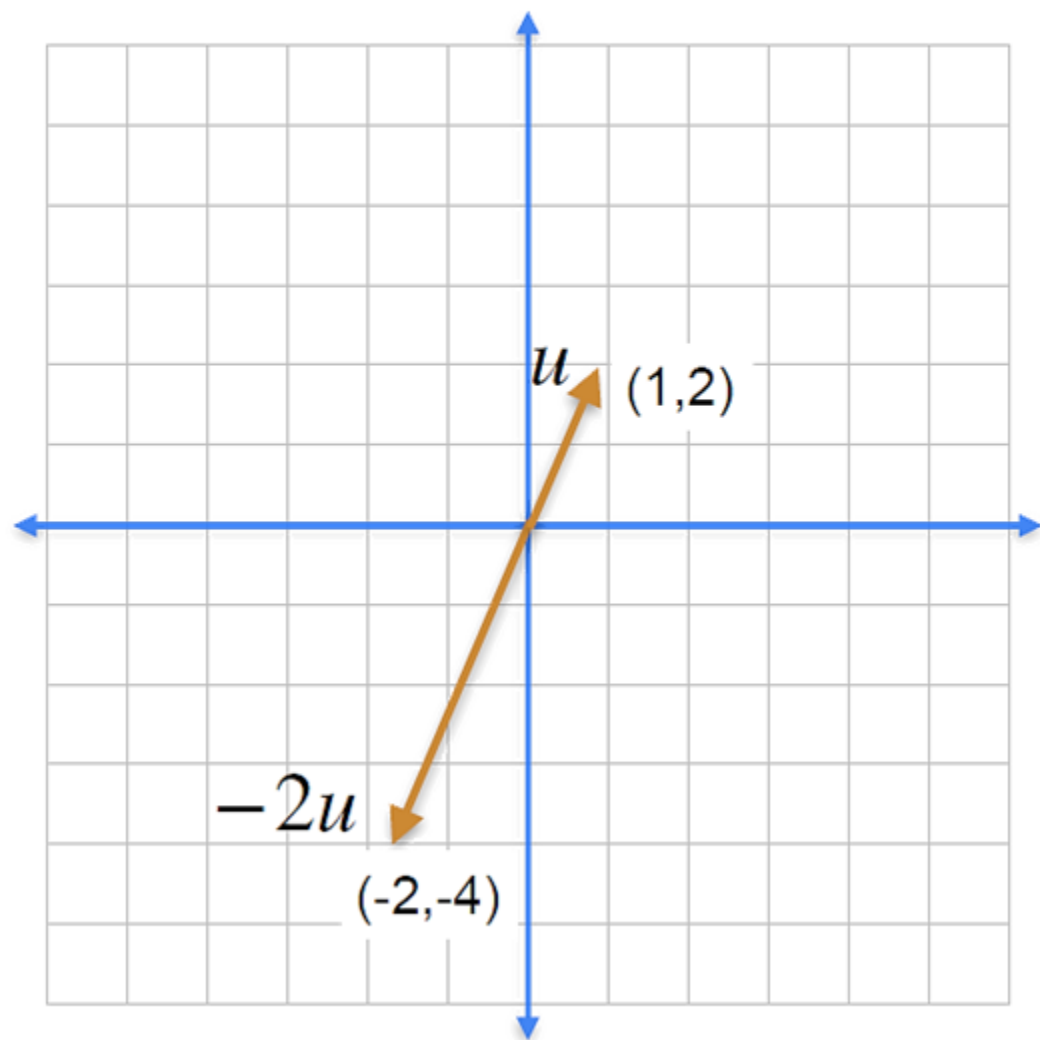


$$u = (1, 2)$$

$$c = 3$$

$$c \times u = (3, 6)$$

Multiplying a Vector by a Scalar



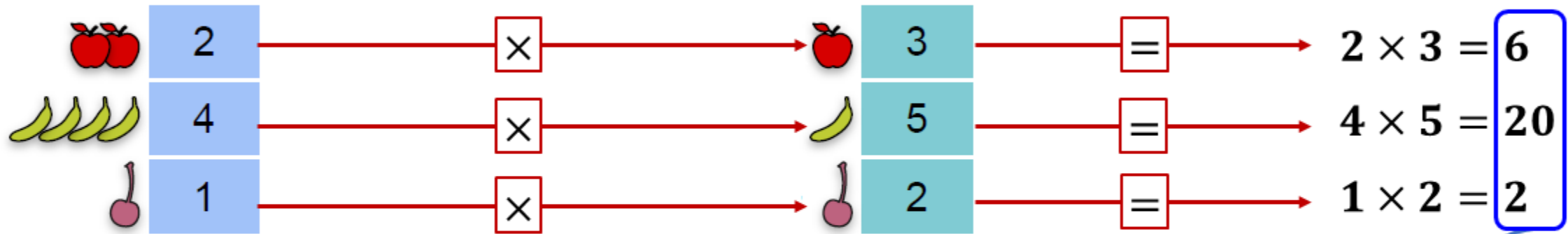
$$u = (1, 2)$$



$$c = -2$$

$$c \times u = (-2, -4)$$

Quantities

Price (*Pound*)Total Price (*Pound*)

28

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

The Dot Product

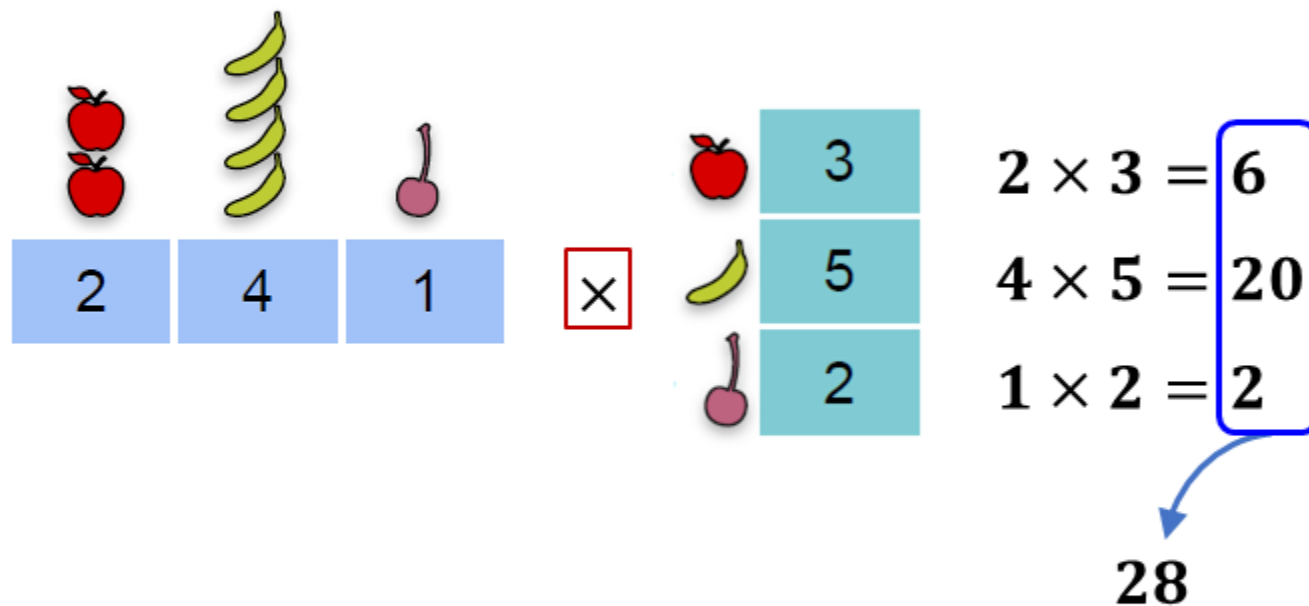
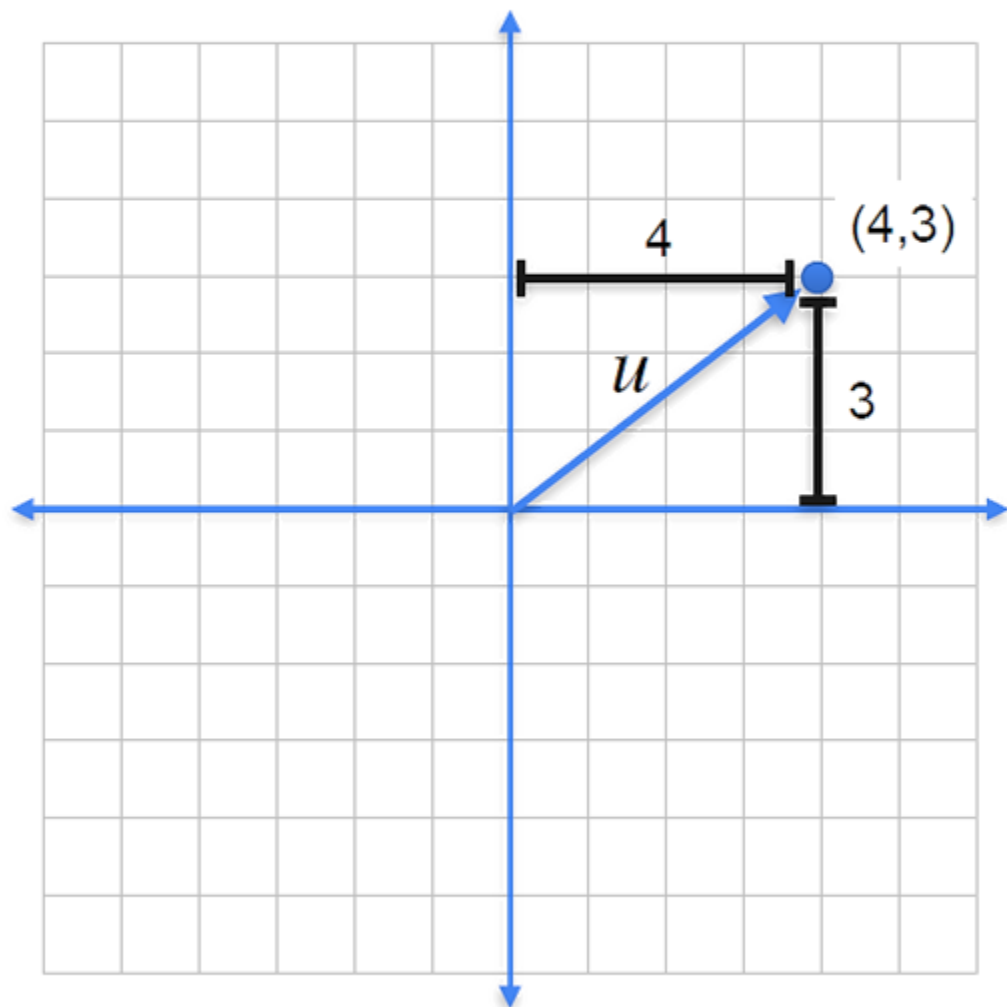


Diagram illustrating the dot product of two vectors using fruit icons:

- Vector 1: 2 Apples, 4 Bananas, 1 Cherry
- Vector 2: 3 Apples, 5 Bananas, 2 Cherries
- Calculation:
$$2 \times 3 = 6$$
$$4 \times 5 = 20$$
$$1 \times 2 = 2$$
- Sum of products: 6 + 20 + 2 = 28

$$[2 \quad 4 \quad 1] \times \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

Norm of a Vector using Dot Product



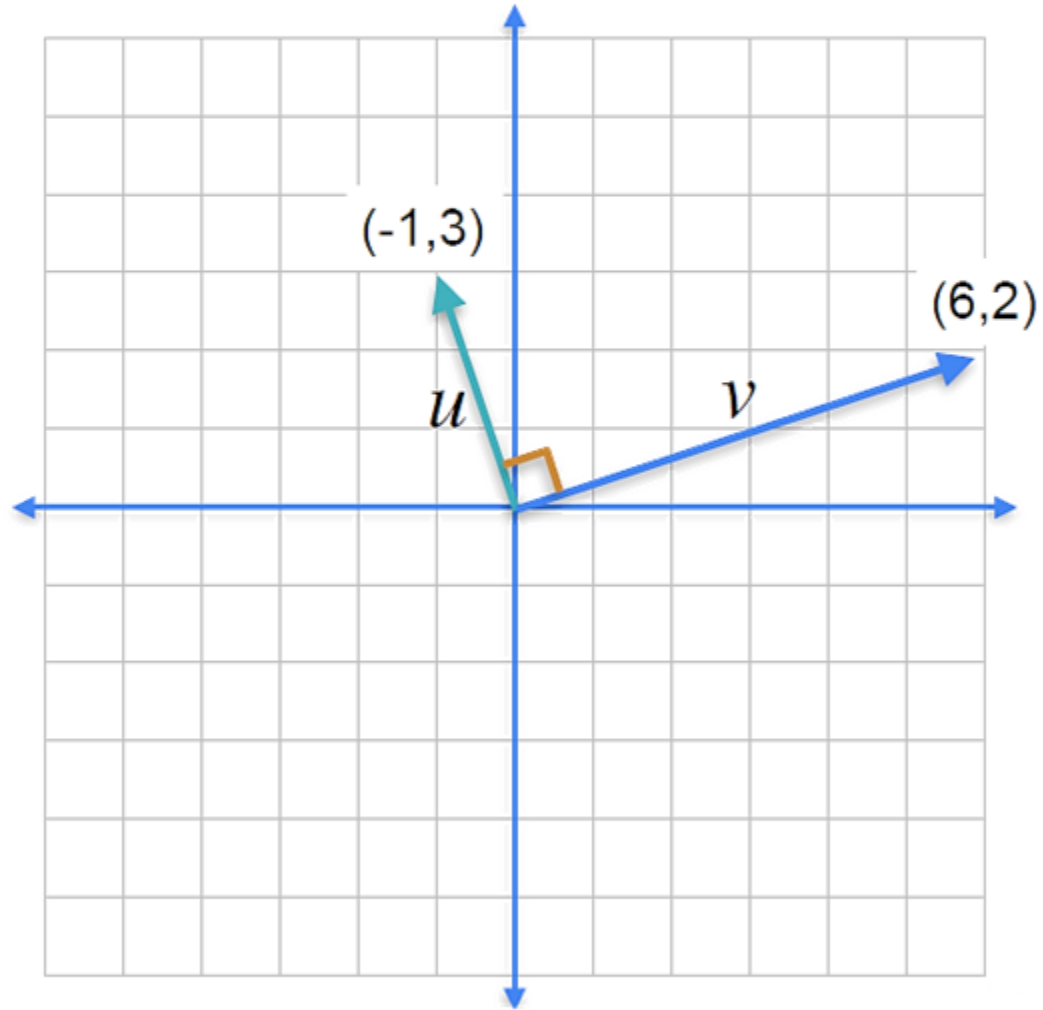
$$\sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\begin{bmatrix} 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 25$$

$$L2 - Norm = \sqrt{\text{Dot Product } (u, u)}$$

$$|u| = \sqrt{\langle u, u \rangle}$$

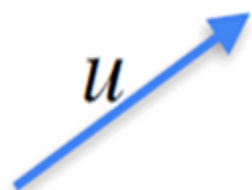
Geometric Dot Product



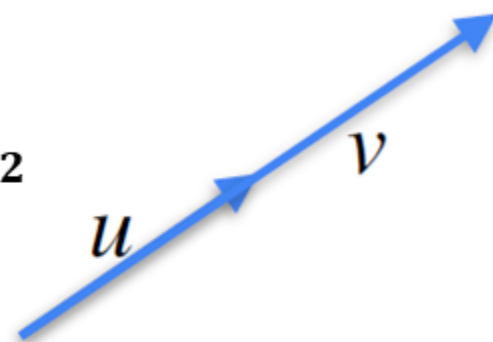
$$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$$

$$\langle u, v \rangle = 0$$

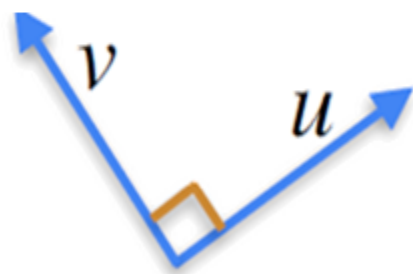
Geometric Dot Product



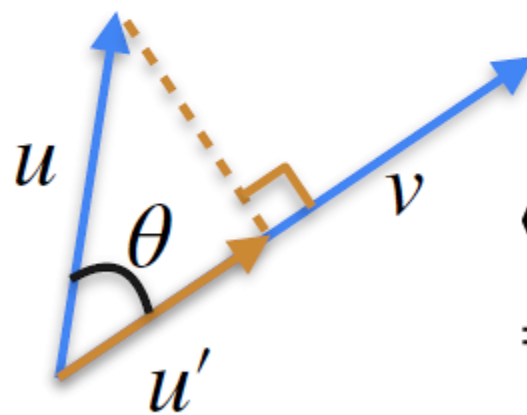
$$\langle u, u \rangle = |u| \times |u| = |u|^2$$



$$\langle u, v \rangle = |u| \times |v|$$

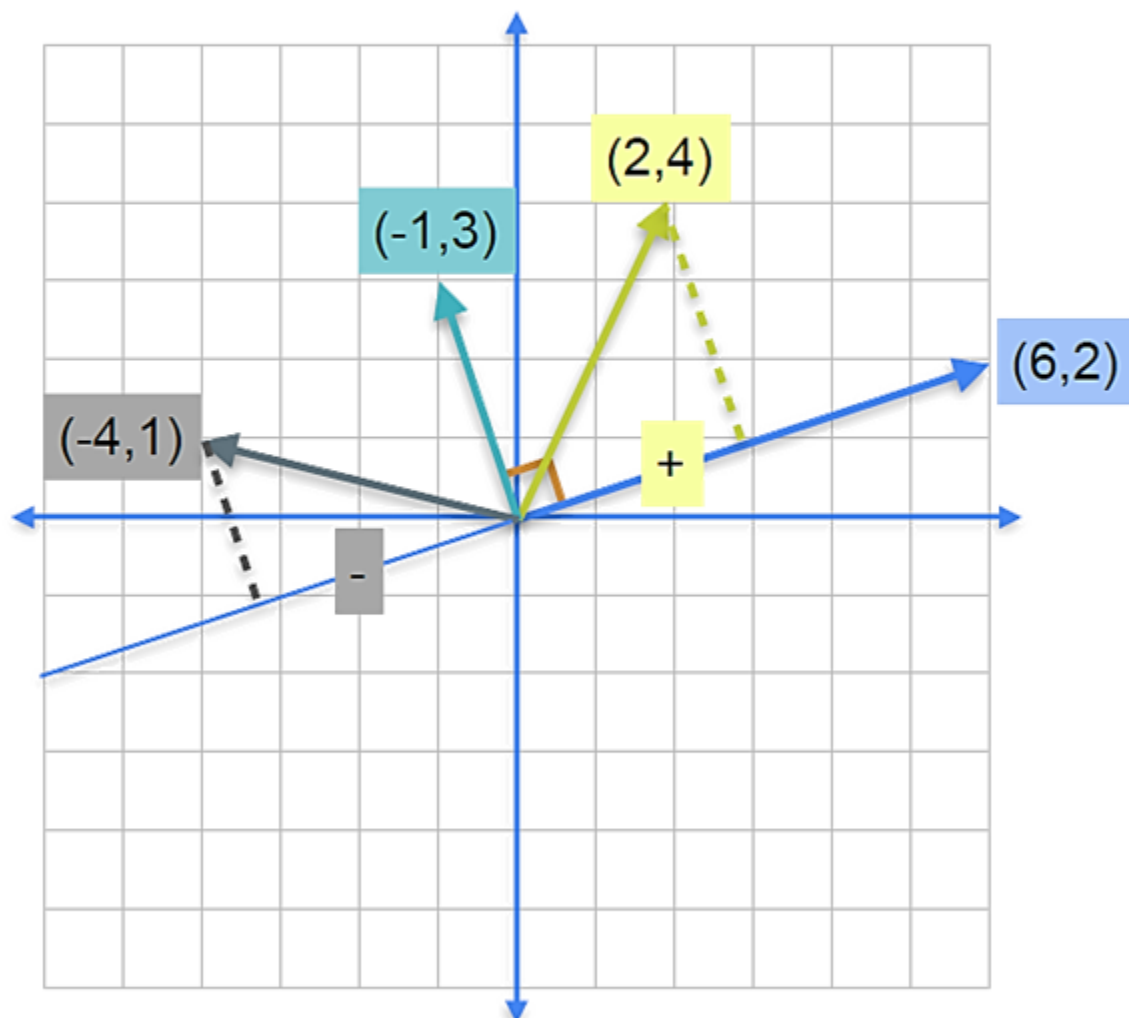


$$\langle u, v \rangle = 0$$



$$\begin{aligned}\langle u, v \rangle &= |u'| \times |v| \\ &= |u| \times |v| \cos \theta\end{aligned}$$

Geometric Dot Product



$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$$

Positive

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 6 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -22$$

Negative

Geometric Dot Product

Dot product with $u > 0$

$$\langle u, v \rangle > 0$$

Dot product with $u < 0$

$$\langle u, v \rangle < 0$$

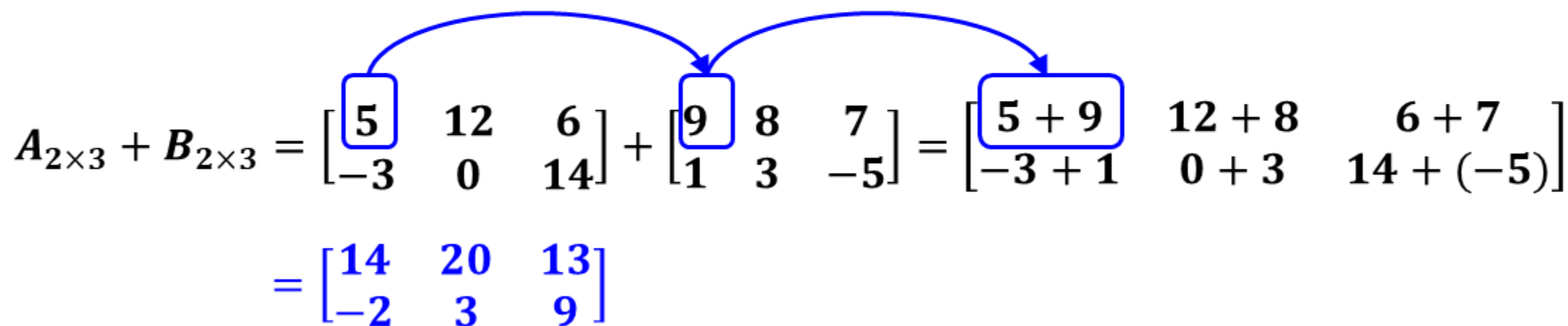
Dot product with $u = 0$

$$\langle u, v \rangle = 0$$

Addition and Subtraction of Matrices

The two matrices must have the same dimensions.

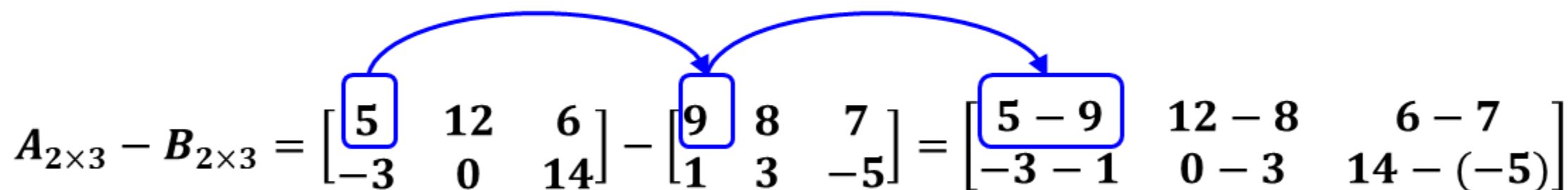
$$A_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 0 & 14 \end{bmatrix} \quad B_{2 \times 3} = \begin{bmatrix} 9 & 8 & 7 \\ 1 & 3 & -5 \end{bmatrix}$$


$$A_{2 \times 3} + B_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 0 & 14 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 1 & 3 & -5 \end{bmatrix} = \begin{bmatrix} 5 + 9 & 12 + 8 & 6 + 7 \\ -3 + 1 & 0 + 3 & 14 + (-5) \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 20 & 13 \\ -2 & 3 & 9 \end{bmatrix}$$

Addition and Subtraction of Matrices

The two matrices must have the same dimensions.

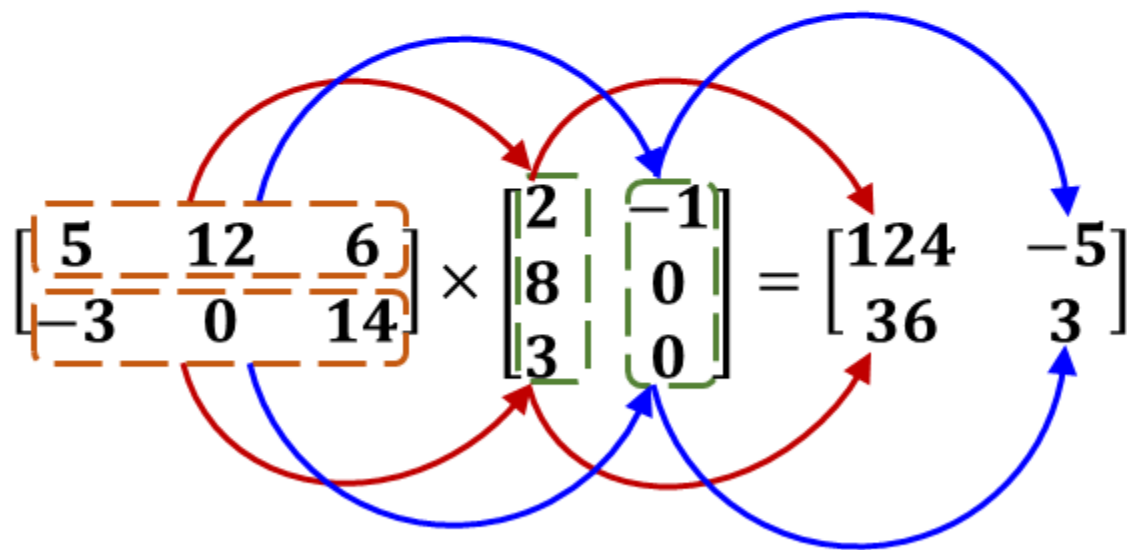
$$A_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 0 & 14 \end{bmatrix} \quad B_{2 \times 3} = \begin{bmatrix} 9 & 8 & 7 \\ 1 & 3 & -5 \end{bmatrix}$$


$$A_{2 \times 3} - B_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 9 & 8 & 7 \\ 1 & 3 & -5 \end{bmatrix} = \begin{bmatrix} 5 - 9 & 12 - 8 & 6 - 7 \\ -3 - 1 & 0 - 3 & 14 - (-5) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 4 & -1 \\ -4 & -3 & 19 \end{bmatrix}$$

Using the same concept, the dot product, we can multiply two matrices. We can only multiply a matrix of dimensions $m \times n$ with a matrix of dimensions $n \times k$.

The second dimension of the first matrix must match the first dimension of the second matrix.

1. *Matrices are nothing more than a collection of vectors.*
2. *When we have a dot product, multiply a row vector times a column vector.*



$$\begin{bmatrix} 5 & 12 & 6 \\ -3 & 0 & 14 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 8 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 124 & -5 \\ 36 & 3 \end{bmatrix}$$

The transpose of a matrix is found by interchanging its rows into columns or columns into rows. The transpose of the matrix is denoted by using the letter “T” in the superscript of the given matrix.

$$A = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \longrightarrow A^T = [3 \quad 4 \quad 5]^T$$

$$A_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 1 & 14 \end{bmatrix} \longrightarrow A_{3 \times 2}^T = \begin{bmatrix} 5 & -3 \\ 12 & 1 \\ 6 & 14 \end{bmatrix}^T$$

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$$A_{2 \times 3} = \begin{bmatrix} 5 & 12 & 6 \\ -3 & 1 & 14 \end{bmatrix} \longrightarrow A_{3 \times 2}^T = \begin{bmatrix} 5 & -3 \\ 12 & 1 \\ 6 & 14 \end{bmatrix}^T$$

***The Tensor** is the most general concept. Scalars, vectors, and matrices are all tensors of ranks zero, one, and two respectively.*

$$[a_{11}]$$

Tensor (1×1)

Rank Zero

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

Tensor ($m \times 1$)

Rank One

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Tensor ($m \times n$)

Rank Two

Tensor ($m \times n \times k$)

Rank Three

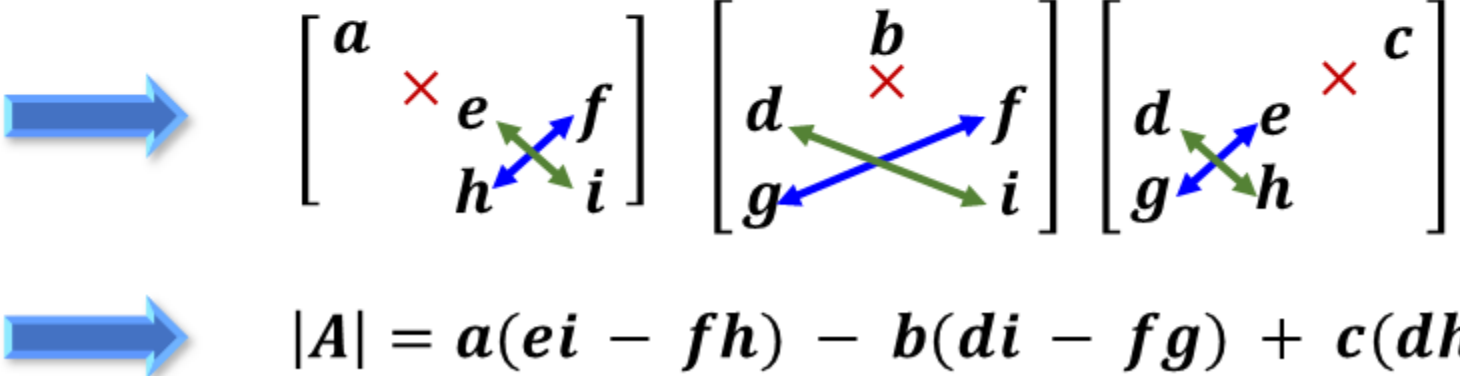
The identity matrix is the matrix that when multiplied by any other matrix it gives the same matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

***The determinant** is a special number that can be calculated from a matrix. The matrix has to be square (same number of rows and columns).*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = ad - bc$$


$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{bmatrix} a & \times e & f \\ & h & i \end{bmatrix} \begin{bmatrix} & b & \times \\ d & & f \\ g & & i \end{bmatrix} \begin{bmatrix} & & c & \times \\ d & e & \\ g & h & \end{bmatrix}$$

$$\Rightarrow |A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$


$$A = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$$

$$|A| = ad - bc$$

$$|A| = 4 \times 8 - 6 \times 3$$

$$|A| = 32 - 18$$

$$|A| = 14$$

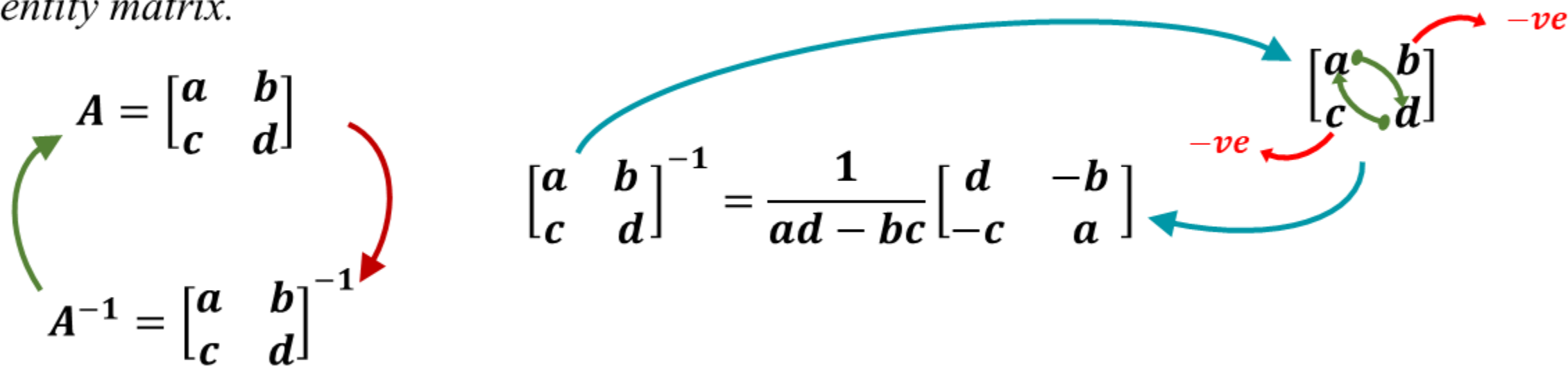
$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$|A| = 6 \times (-54) - 1 \times (18) + 1 \times (36)$$

$$|A| = -360$$

Inverse of a Matrix is a matrix that on multiplying with the original matrix result in the identity matrix.



The diagram illustrates the process of finding the inverse of a 2x2 matrix. On the left, a green curved arrow points from the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to its inverse $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$. A red curved arrow points from A^{-1} back to A . In the center, the formula for the inverse is given: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. On the right, a small 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is shown with green arrows indicating a swap between a and d , and red arrows indicating the negation of b and c (labeled $-ve$). A large blue curved arrow connects the original matrix A to the inverse matrix formula.

In other words: swap the positions of **a** and **d**, put negatives in front of **b** and **c**, and divide everything by **ad - bc** (the determinant of the matrix)

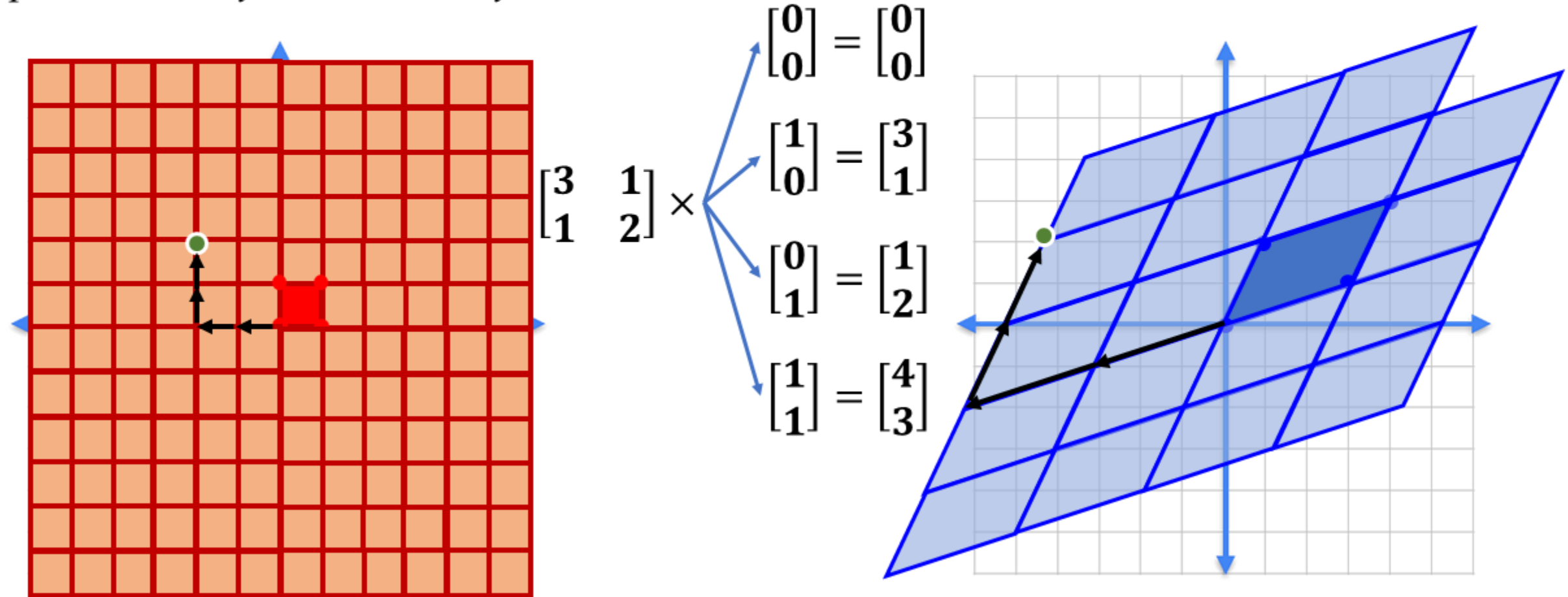
$$\begin{aligned}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} &= \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}\end{aligned}$$

How do we know this is the right answer?

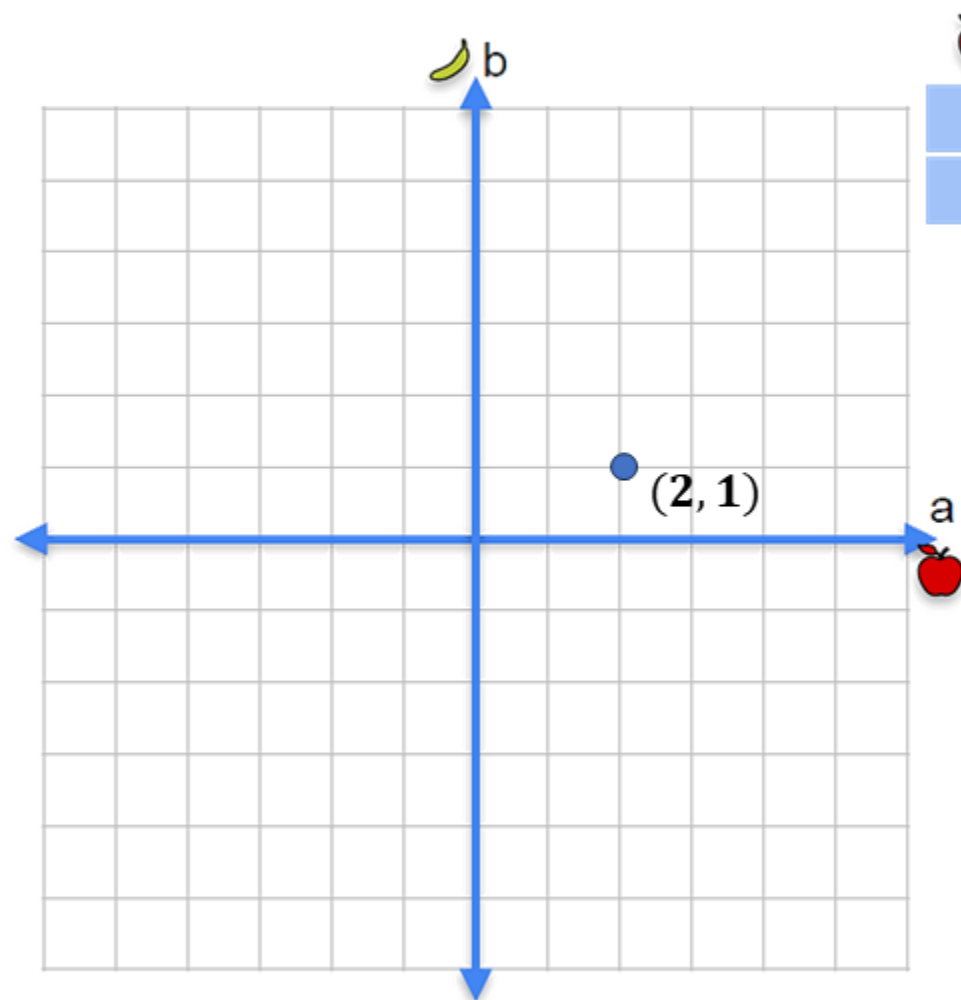
$$\begin{aligned}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 4 \times 0.6 + 7 \times -0.2 & 4 \times -0.7 + 7 \times 0.4 \\ 2 \times 0.6 + 6 \times -0.2 & 2 \times -0.7 + 6 \times 0.4 \end{bmatrix} \\ &= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

Non-zero determinants mean that the matrix has an inverse and a zero determinant, that the matrix does not have an inverse.

*A **linear transformation** is a way to send each point in the plane into another point in the plane in a very structured way.*



A linear Transformation



		
3	1	2
1	2	1

 $\times$ $=$

Day 1
Day 2

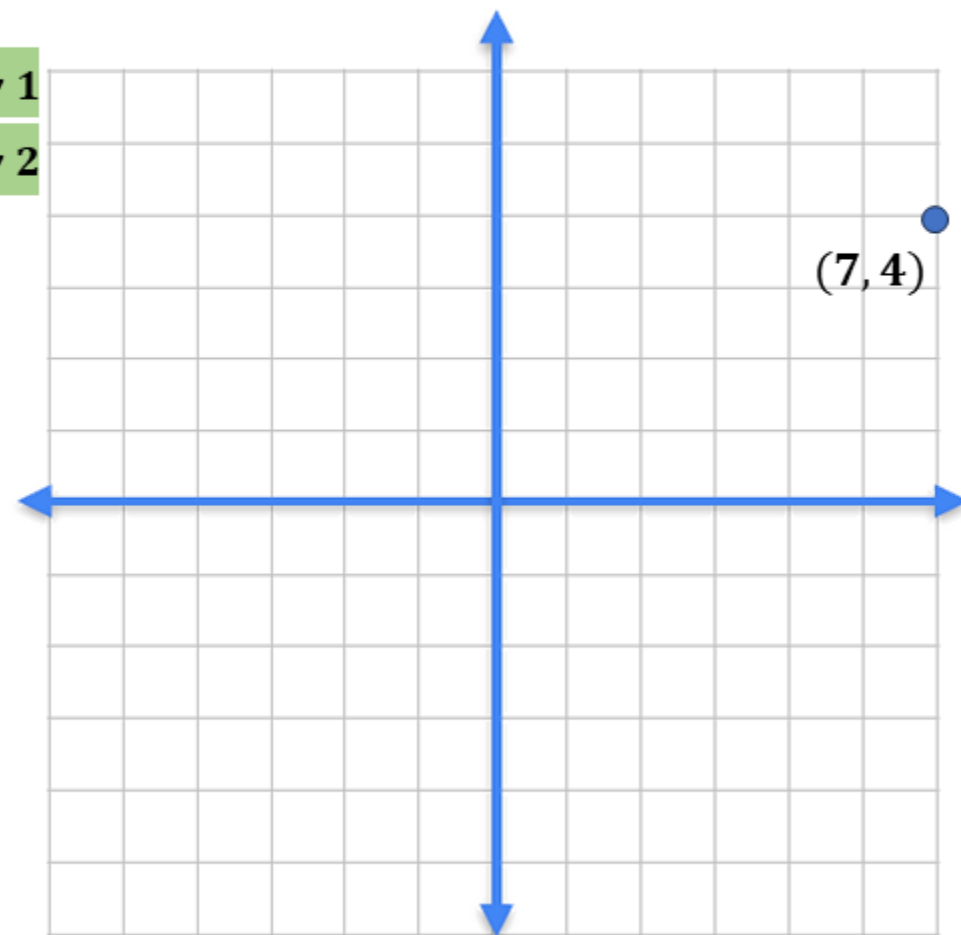
$$\text{Day 1} = 3a + b$$

$$\text{Day 2} = a + 2b$$

$$a = 2$$

$$b = 1$$

Day 1	7
Day 2	4



Spam Filter

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2

Lottery: 3 point

Win: 2 points

“Win, win the lottery!”: 7points

Rule:

If the number of points of the sentence is bigger than (*Threshold*), then the email is spam.

Goal: Find the best points and threshold

Lottery: ? point

Win: ? point

Threshold: ? points

Spam Filter

Spam	Lottery	Win	Score	> 1.5?
Yes	1	1	2	Yes
Yes	2	1	3	Yes
No	0	0	0	No
Yes	0	2	2	Yes
No	0	1	1	No
No	1	0	1	No
Yes	2	2	4	Yes
Yes	2	0	2	Yes
Yes	1	2	3	Yes

Solution**Lottery:** 1 point**Win:** 1 point**Threshold:** 1.5 points

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2

*Spam Filter***Line:**

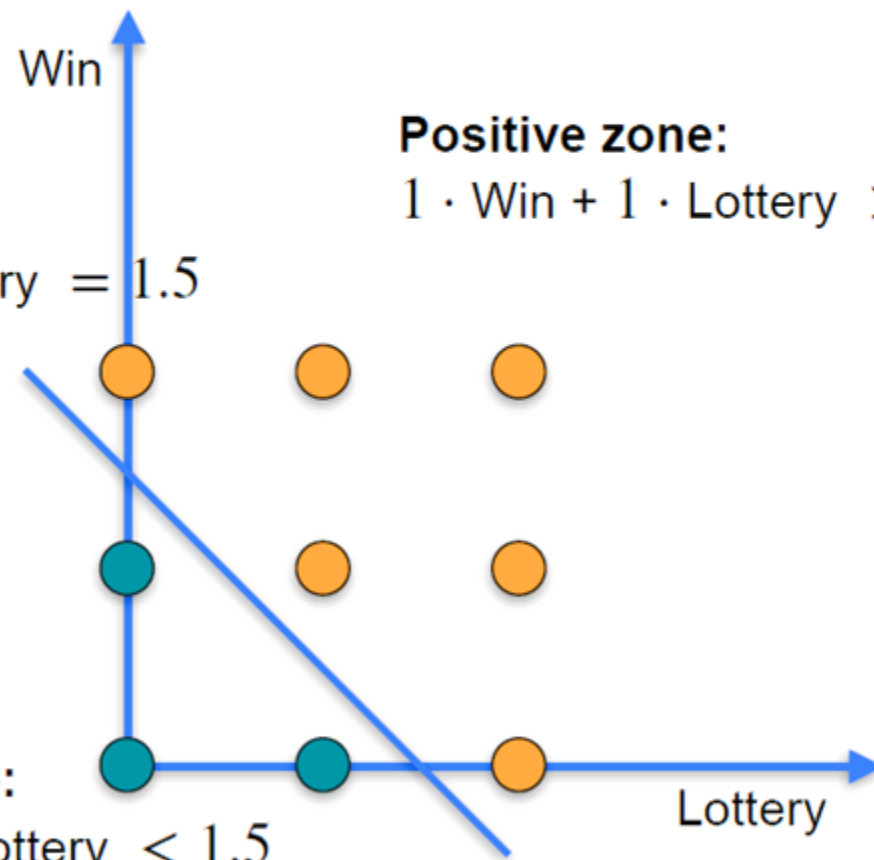
$$1 \cdot \text{Win} + 1 \cdot \text{Lottery} = 1.5$$

Positive zone:

$$1 \cdot \text{Win} + 1 \cdot \text{Lottery} > 1.5$$

Negative zone:

$$1 \cdot \text{Win} + 1 \cdot \text{Lottery} < 1.5$$



Spam Filter

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2

2	1
---	---

$$2 \times 1 + 1 \times 1 = 2 + 1 = 3$$

Model
1
1

= 3

Check: > 1.5?



Spam

Spam Filter

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2



$$0 \times 1 + 1 \times 1 = 0 + 1 = 1$$

Model
1
1

 $= 1$ Check: > 1.5 ?

Not spam

Spam Filter

Spam	Lottery	Win
Yes	1	1
Yes	2	1
No	0	0
Yes	0	2
No	0	1
No	1	0
Yes	2	2
Yes	2	0
Yes	1	2

Model
1
1

=

Prod
2
3
0
2
1
1
4
2
3

Check: >1.5?



Check
Yes
Yes
No
Yes
No
No
Yes
Yes
Yes

Spam Filter

Spam	Lottery	Win	Bias
Yes	1	1	1
Yes	2	1	1
No	0	0	1
Yes	0	2	1
No	0	1	1
No	1	0	1
Yes	2	2	1
Yes	2	0	1
Yes	1	2	1

Check

$$1 \cdot \text{Win} + 1 \cdot \text{Lottery} - 1.5 > 0$$

$$1 \cdot \text{Win} + 1 \cdot \text{Lottery} - 1.5 > 0$$

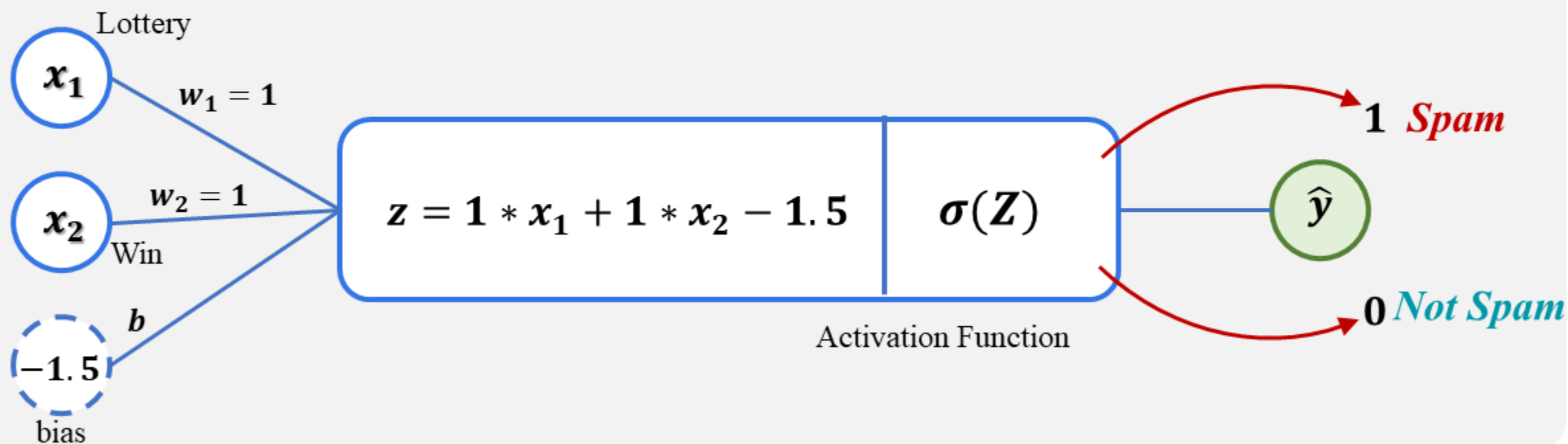
Threshold

Bias

Model
1
1
-1.5

Check: > 1.5 ?Check: > 0 ?

Bias

Spam Filter

Digital images is made up of tiny dots called pixels. Each pixel in an image is made up of binary numbers which is the color depth of the image.

 400×400

=



+



+

 $3 (400 \times 400)$ **Tensors**



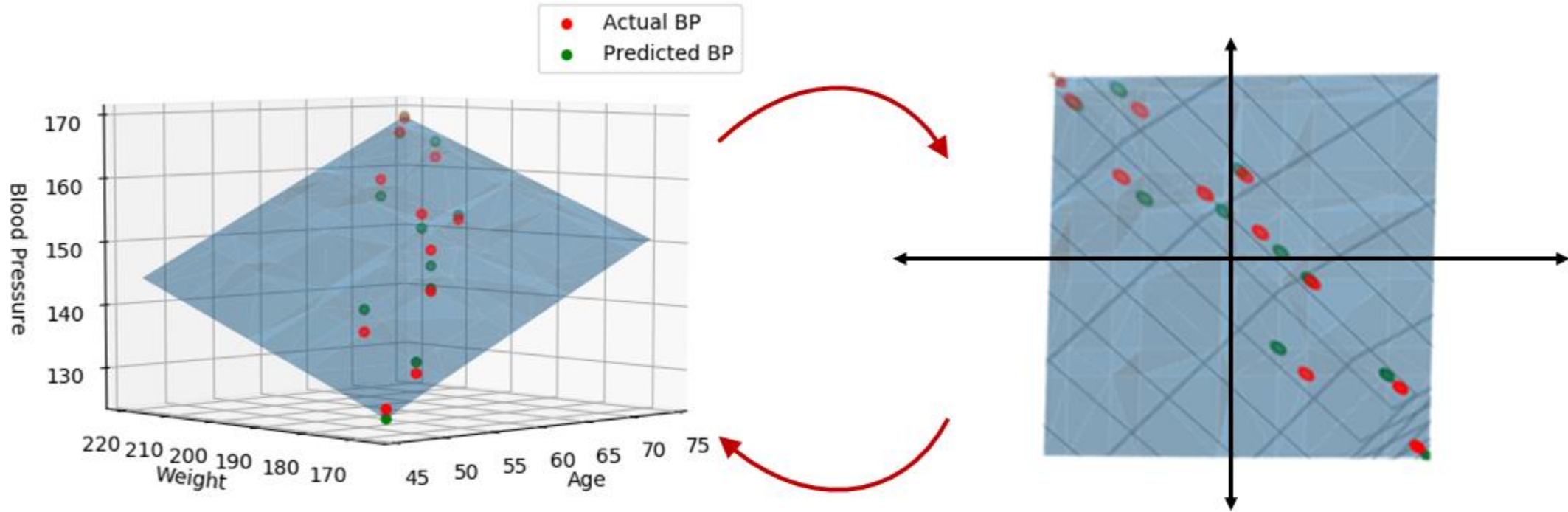
mnist



cifar 10



cifar 100

**Transformation**

