

Mathematics

Lecture 7

MAT 122

System of Linear Equations

- Solving System of Linear Equations
- Cremer's Method Definition
- Cremer's Method 2×2 matrix
- Cremer's Method 3×3 matrix
- Gaussian Elimination Method
- Extra Practice

The need to solve systems of linear equations arises frequently in engineering. The analysis of electric circuits and the control of systems are two examples.

Cramer's rule for solving such systems involves the calculation of determinants and their ratio. It is a useful solution for systems containing only a few equations.

In linear algebra, **Cramer's rule** is a specific formula used for solving a system of linear equations containing as many equations as unknowns, efficient whenever the system of equations has a unique solution. This rule is named after Gabriel Cramer (1704–1752), who published the rule for an arbitrary number of unknowns in 1750.

This is the most commonly used formula for getting the solution for the given system of equations formed through matrices. The solution obtained using Cramer's rule will be in terms of the determinants of the coefficient matrix and matrices obtained from it by replacing one column with the column vector of the right-hand sides of the equations.

Cramer's rule is one of the important methods applied to solve a system of equations. In this method, the values of the variables in the system are to be calculated using the determinants of matrices. Thus, Cramer's rule is also known as the determinant method.

Cramer's Rule Formula

Consider a system of linear equations with n variables $x_1, x_2, x_3, \dots, x_n$ written in the matrix form $AX = B$.

Here,

A = Coefficient matrix (must be a square matrix)

X = Column matrix with variables

B = Column matrix with the constants (which are on the right side of the equations)

Cramer's rule for the 2×2 matrix is applied to solve the system of equations in two variables.

Let us consider two linear equations in two variables.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Let us write these two equations in the form of $AX = B$.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here,

$$\text{Coefficient matrix} = A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$\text{Variable matrix} = X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{Constant matrix} = B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

And

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

Therefore,

$$x = D_x/D$$

$$y = D_y/D$$

Solve the following system of equations using Cramer's rule:

$$2x - y = 5$$

$$x + y = 4$$

$$2x - y = 5$$

$$x + y = 4$$

Let us write these equations in the form $AX = B$.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$D = |A|$$

$$= \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2(1) - (-1)(1)$$

$$= 2 + 1$$

$$= 3 \neq 0$$

So, the given system of equations has a unique solution.

$$D_x = \begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}$$

$$= 5(1) - (-1)(4)$$

$$= 5 + 4$$

$$= 9$$

$$\begin{aligned} D_y &= \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} \\ &= 2(4) - 5(1) \\ &= 8 - 5 \\ &= 3 \end{aligned}$$

Therefore,

$$x = D_x/D = 9/3 = 3$$

$$y = D_y/D = 3/3 = 1$$

To find the Cramer's rule formula for a 3×3 matrix, we need to consider the system of 3 equations with three variables.

Consider:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let us write these equations in the form $AX = B$.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Therefore, $x = D_x/D$, $y = D_y/D$, $z = D_z/D$; $D \neq 0$

Question:

Solve the following system of equations using Cramer's rule:

$$x + y + z = 6$$

$$y + 3z = 11$$

$$x - 2y + z = 0$$

Solution:

Given,

$$x + y + z = 6$$

$$y + 3z = 11$$

$$x - 2y + z = 0$$

Let us write these equations in the form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = 1(1 + 6) - 1(0 - 3) + 1(0 - 1) = 7 + 3 - 1 = 9$$

$D \neq 0$ so the given system of equations has a unique solution.

Also,

$$D_x = \begin{vmatrix} 6 & 1 & 1 \\ 11 & 1 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 6(1 + 6) - 1(11 - 0) + 1(-22 - 0) = 42 - 11 - 22 = 9$$

$$D_y = \begin{vmatrix} 1 & 6 & 1 \\ 0 & 11 & 3 \\ 1 & 0 & 1 \end{vmatrix} = 1(11 - 0) - 6(0 - 3) + 1(0 - 11) = 11 + 18 - 11 = 18$$

$$D_z = \begin{vmatrix} 1 & 1 & 6 \\ 0 & 1 & 11 \\ 1 & -2 & 0 \end{vmatrix} = 1(0 + 22) - 1(0 - 11) + 6(0 - 1) = 22 + 11 - 6 = 27$$

$$x = D_x/D = 9/9 = 1$$

$$y = D_y/D = 18/9 = 2$$

$$z = D_z/D = 27/9 = 3$$

Gaussian Elimination Method

In mathematics, **Gaussian elimination**, also known as **row reduction**, is an algorithm for solving systems of linear equations. It consists of a sequence of operations performed on the corresponding matrix of coefficients. This method can also be used to compute the rank of a matrix, the determinant of a square matrix, and the inverse of an invertible matrix.

In mathematics, the Gaussian elimination method is known as the row reduction algorithm for solving linear equations systems. It consists of a sequence of operations performed on the corresponding matrix of coefficients.

We can also use this method to estimate either of the following:

- The rank of the given matrix
- The determinant of a square matrix
- The inverse of an invertible matrix

Question:

Solve the following system of equations:

$$x + y + z = 2$$

$$x + 2y + 3z = 5$$

$$2x + 3y + 4z = 11$$

Solution:

Given system of equations are:

$$x + y + z = 2$$

$$x + 2y + 3z = 5$$

$$2x + 3y + 4z = 11$$

Let us write these equations in matrix form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 11 \end{array} \right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right]$$

No Solution !!!!!

Let's Practice

Use Cramers rule to solve the system of 2 linear equations for 2 unknowns as shown below:

$$\begin{aligned}x + 2y &= 3 \\ 2x + 3y &= 1\end{aligned}$$

$$D = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (1)(3) - (2)(2) = 3 - 4 = -1$$

$$D_x = \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = (3)(3) - (1)(2) = 9 - 2 = 7$$

$$D_y = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(3) = 1 - 6 = -5$$

To finally find the unique solution for each variable in the equations, we now solve the Cramers rule equation for each of them:

$$x = \frac{D_x}{D} = \frac{7}{-1} = -7$$

$$y = \frac{D_y}{D} = \frac{-5}{-1} = 5$$

Equation 10: Solution to the variables x and y

And so, the system has been solved! The solutions for the variables x and y are: $x = -7, y = 5$.

Now let us take a look at the process on how to do the Cramers rule 3x3!

Use Cramers rule to solve the system of equations for 3 unknowns as shown below:

$$x + 4y + 3z = 1$$

$$x + 2y + 9z = 1$$

$$x + 6y + 6z = 1$$

And then we calculate the 4 required determinants. Starting with determinant D :

$$D = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 9 \\ 1 & 6 & 6 \end{vmatrix} = 1\{(2)(6) - (6)(9)\} - 4\{(1)(6) - (1)(9)\} + 3\{(1)(6) - (1)(2)\}$$

$$= \{12 - 54\} - 4\{6 - 9\} + 3\{6 - 2\} = -42 - 4(-3) + 3(4)$$

$$= -42 + 12 + 12 = -42 + 24$$

$$= -18$$

This determinant D is just the determinant of the square coefficient matrix from the left hand side of the augmented matrix in equation 12. Now, to solve the determinants related to each of the variables in the equations from the system we substitute the column related to each variable in the square coefficient matrix by the column coming from the right hand side of the augmented matrix. And so, the determinants D_x , D_y , D_z are:

$$D_x = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 2 & 9 \\ 1 & 6 & 6 \end{vmatrix} = 1\{(2)(6) - (6)(9)\} - 4\{(1)(6) - (1)(9)\} + 3\{(1)(6) - (1)(2)\}$$

$$= \{12 - 54\} - 4\{6 - 9\} + 3\{6 - 2\} = -42 - 4(-3) + 3(4)$$

$$= -42 + 12 + 12 = -42 + 24$$

$$= -18$$

$$D_y = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 9 \\ 1 & 1 & 6 \end{vmatrix} = 1\{(1)(6) - (1)(9)\} - 1\{(1)(6) - (1)(9)\} + 3\{(1)(1) - (1)(1)\}$$

$$= \{6 - 9\} - \{6 - 9\} + 3\{0\}$$

$$= 0$$

$$D_z = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 2 & 1 \\ 1 & 6 & 1 \end{vmatrix} = 1\{(2)(1) - (6)(1)\} - 4\{(1)(1) - (1)(1)\} + 1\{(1)(6) - (1)(2)\}$$

$$= \{2 - 6\} - 4\{0\} + \{6 - 2\} = -4 + 0 + 4$$

$$= 0$$

Solving for the variables x , y and z :

$$x = \frac{D_x}{D} = \frac{-18}{-18} = 1$$

$$y = \frac{D_y}{D} = \frac{0}{-18} = 0$$

$$z = \frac{D_z}{D} = \frac{0}{-18} = 0$$

Equation 15: Solution to the variables x and y

And so, the system has been solved! $x = 1, y = 0, z = 0$

Solve the following linear system using Cramers rule:

$$5x + 3y = 1$$

$$x + y = 2$$

Equation 16: System of 2 linear equations for 2 unknowns

And then we calculate the 3 required determinants:

$$D = \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix} = (5)(1) - (1)(3) = 5 - 3 = 2$$

$$D_x = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(3) = 1 - 6 = -5$$

$$D_y = \begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} = (5)(2) - (1)(1) = 10 - 1 = 9$$

Now use D , D_x , D_y to finally find the unique solution for each variable in the equations:

$$x = \frac{D_x}{D} = \frac{-5}{2} = -2.5$$

$$y = \frac{D_y}{D} = \frac{9}{2} = 4.5$$

So, the solutions for the variables x and y are: $x = -2.5$, $y = 4.5$

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