

Probability and Statistics

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Lecture 8

جامعة الجميع الذكية

تسعى لتكون الكيان الأول الذي يتبنى مبدأ المجتمع التعليمي
التعاوني ويقدم العلم مقابل العمل لتنمية وبناء المجتمعات.

STA 201

Hypothesis Testing with Two Samples



جامعة الجميع الذكية
EVERYONE'S SMART UNIVERSITY

8.1 Testing the Difference Between Means (Large Independent Samples)

8.2 Testing the Difference Between Means (Small Independent Samples)

8.3 Testing the Difference Between Means (Dependent Samples)

8.4 Testing the Difference Between Proportions

8.1 Testing the Difference Between Means (Large Independent Samples)

- Determine whether two samples are independent or dependent
- Perform a two-sample z-test for the difference between two means μ_1 and μ_2 using large independent samples

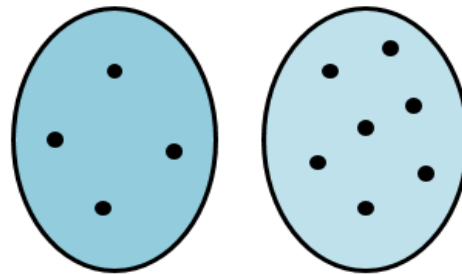
Independent Samples

The sample selected from one population is not related to the sample selected from the second population.

Dependent Samples (paired or matched samples)

Each member of one sample corresponds to a member of the other sample.

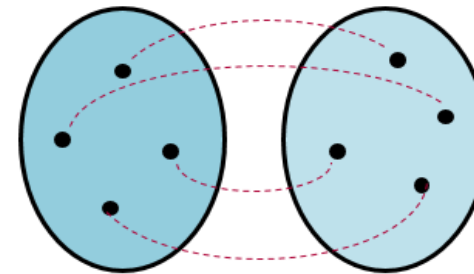
Independent Samples



Sample 1

Sample 2

Dependent Samples



Sample 1

Sample 2

Classify the pair of samples as independent or dependent.

- Sample 1: Resting heart rates of 35 individuals before drinking coffee.
- Sample 2: Resting heart rates of the same individuals after drinking two cups of coffee.

Solution:

Dependent Samples (The samples can be paired with respect to each individual)

Classify the pair of samples as independent or dependent.

- Sample 1: Test scores for 35 statistics students.
- Sample 2: Test scores for 42 biology students who do not study statistics.

Solution:

Independent Samples (Not possible to form a pairing between the members of the samples; the sample sizes are different, and the data represent scores for different individuals.)

1. Null hypothesis H_0

A statistical hypothesis that usually states there is no difference between the parameters of two populations.

Always contains the symbol $\leq$, $=$, or $\geq$.

2. Alternative hypothesis H_a

A statistical hypothesis that is true when H_0 is false.

Always contains the symbol $>$, $\neq$, or $<$.

$$\begin{cases} H_0: \mu_1 = \mu_2 \\ H_a: \mu_1 \neq \mu_2 \end{cases} \quad \begin{cases} H_0: \mu_1 \leq \mu_2 \\ H_a: \mu_1 > \mu_2 \end{cases} \quad \begin{cases} H_0: \mu_1 \geq \mu_2 \\ H_a: \mu_1 < \mu_2 \end{cases}$$

Regardless of which hypotheses you use, you always assume there is no difference between the population means, or $\mu_1 = \mu_2$.

Two Sample z-Test for the Difference Between Means Large Samples

Three conditions are necessary to perform a z-test for the difference between two population means μ_1 and μ_2 .

1. The samples must be randomly selected.
2. The samples must be independent.
3. Each sample size must be at least 30, or, if not, each population must have a normal distribution with a known standard deviation.

Steps for conducting Testing the Difference Between Means (Large Independent Samples)

1- State H_0 and H_a

2- identify α

3- Critical value

4- Test statistic

5- Take decision

The **test statistic** is

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

When the samples are large, you can use s_1 and s_2 in place of σ_1 and σ_2 . If the samples are not large, you can still use a two-sample z-test, provided the populations are normally distributed and the population standard deviations are known.

Example: Two-Sample z-Test for the Difference Between Means

A consumer education organization claims that there is a difference in the mean credit card debt of males and females in the United States. The results of a random survey of 200 individuals from each group are shown below. The two samples are independent. Do the results support the organization's claim? Use $\alpha = 0.05$.



Females (1)	Males (2)
$\bar{x}_1 = \$2290$	$\bar{x}_2 = \$2370$
$s_1 = \$750$	$s_2 = \$800$
$n_1 = 200$	$n_2 = 200$



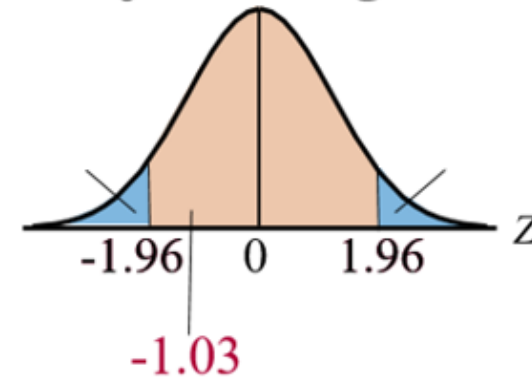
Solution: Two-Sample z-Test for the Difference Between Means

- $H_0: \mu_1 = \mu_2$
- $H_a: \mu_1 \neq \mu_2$
- $\alpha = 0.05$
- $n_1 = 200, n_2 = 200$
- critical value = -1.96 and 1.96
- **Test Statistic:**

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$z = \frac{(2290 - 2370) - 0}{\sqrt{\frac{750^2}{200} + \frac{800^2}{200}}} = -1.03$$

- **Rejection Region:**



- **Decision: Fail to Reject H_0**
At the 5% level of significance, there is not enough evidence to support the organization's claim that there is a difference in the mean credit card debt of males and females.

8.2 Testing the Difference Between Means (Small Independent Samples)

If samples of size less than 30 are taken from normally-distributed populations, a t -test may be used to test the difference between the population means μ_1 and μ_2 .

Three conditions are necessary to use a t -test for small independent samples:

1. The samples must be randomly selected.
2. The samples must be independent.
3. Each population must have a normal distribution.

Testing the Difference Between Means (Small Independent Samples)

The **standardized test statistic** is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$$

The standard error and the degrees of freedom of the sampling distribution depend on whether the population variances are equal.

Testing the Difference Between Means (Small Independent Samples)

- **Variances are equal**

- The standard error for the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

- d.f. = $n_1 + n_2 - 2$

- **Variances are not equal**

- the standard error is

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- d.f = smaller of $n_1 - 1$ or $n_2 - 1$

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_{\bar{x}_1 - \bar{x}_2}}$$



- **Variances are equal**

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- d.f. = $n_1 + n_2 - 2$

- **Variances are not equal**

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- d.f = smaller of $n_1 - 1$ or $n_2 - 1$

Example: Two-Sample t -Test for the Difference Between Means

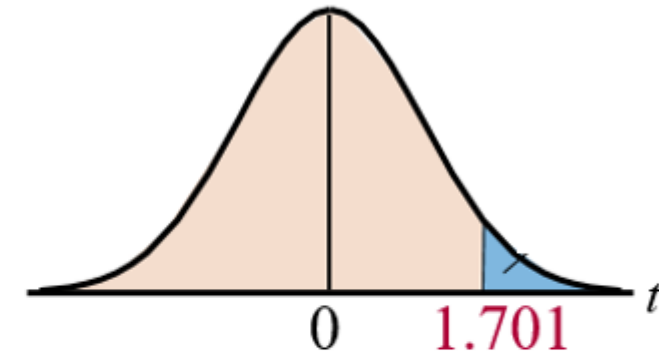
A manufacturer claims that the calling range (in feet) of its 2.4-GHz cordless telephone is greater than that of its leading competitor. You perform a study using 14 randomly selected phones from the manufacturer and 16 randomly selected similar phones from its competitor. The results are shown below. At $\alpha = 0.05$, can you support the manufacturer's claim? Assume the populations are normally distributed and the population variances are equal.

Manufacturer (1)	Competition (2)
$\bar{x}_1 = 1275 \text{ ft}$	$\bar{x}_2 = 1250 \text{ ft}$
$s_1 = 45 \text{ ft}$	$s_2 = 30 \text{ ft}$
$n_1 = 14$	$n_2 = 16$

Solution: Two-Sample t -Test for the Difference Between Means

- $H_0: \mu_1 \leq \mu_2$
- $H_a: \mu_1 > \mu_2$
- $\alpha = 0.05$
- $\text{d.f.} = 14 + 16 - 2 = 28$
- critical value = 1.701
- Test Statistic: ?

- Rejection Region:



- Decision: ?

Solution: Obtaining Test statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

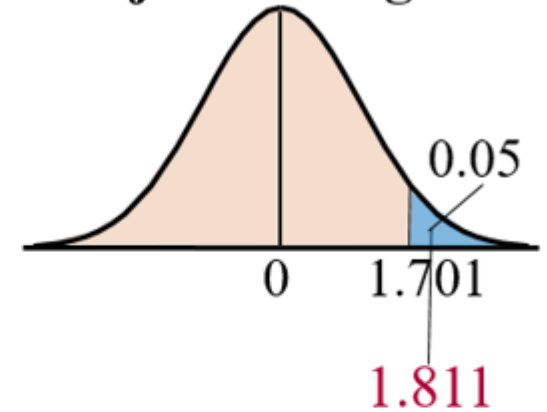
$$t = \frac{(1275 - 1250) - 0}{\sqrt{\frac{(14 - 1)(45)^2 + (16 - 1)(30)^2}{14 + 16 - 2}} \cdot \sqrt{\frac{1}{14} + \frac{1}{16}}}$$

$$t = \frac{(1275 - 1250) - 0}{13.8018} \approx 1.811$$

Solution: Two-Sample t -Test for the Difference Between Means

- $H_0: \mu_1 \leq \mu_2$
- $H_a: \mu_1 > \mu_2$
- $\alpha = 0.05$
- $\text{d.f.} = 14 + 16 - 2 = 28$
- critical value = 1.701
- **Test Statistic:**
 $t = 1.811$

- **Rejection Region:**



- **Decision: Reject H_0**
At the 5% level of significance, there is enough evidence to support the manufacturer's claim that its phone has a greater calling range than its competitors.

Example: Two-Sample t -Test for the Difference Between Means

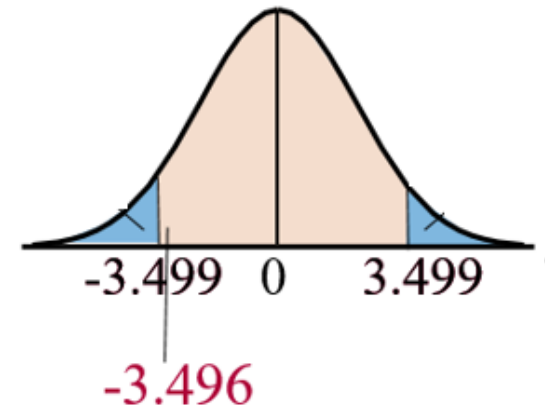
The braking distances of 8 Volkswagen GTIs and 10 Ford Focuses were tested when traveling at 60 miles per hour on dry pavement. The results are shown below. Can you conclude that there is a difference in the mean braking distances of the two types of cars? Use $\alpha = 0.01$. Assume the populations are normally distributed and the population variances are not equal. (*Adapted from Consumer Reports*)

GTI (1)	Focus (2)
$\bar{x}_1 = 134 \text{ ft}$	$\bar{x}_2 = 143 \text{ ft}$
$s_1 = 6.9 \text{ ft}$	$s_2 = 2.6 \text{ ft}$
$n_1 = 8$	$n_2 = 10$

- $H_0: \mu_1 = \mu_2$
- $H_a: \mu_1 \neq \mu_2$
- $\alpha = 0.01$
- $\text{d.f.} = 8 - 1 = 7$
- critical value = -3.499 and 3.499
- **Test Statistic:**

$$t = \frac{(134 - 143) - 0}{\sqrt{\frac{6.9^2}{8} + \frac{2.6^2}{10}}} = -3.496$$

- **Rejection Region:**



- **Decision: Fail to Reject H_0**
At the 1% level of significance, there is not enough evidence to conclude that the mean braking distances of the cars are different.

Section 8.3: Testing the Difference Between Means (Dependent Samples)

- To perform a two-sample hypothesis test with dependent samples, the difference between each data pair is first found:
 - $d = x_1 - x_2$ Difference between entries for a data pair
- The test statistic is the mean $\bar{d}$ of these differences.
 - $\bar{d} = \frac{\sum d}{n}$ Mean of the differences between paired data entries in the dependent samples
- The standard deviation of the differences between the paired data entries in the dependent samples

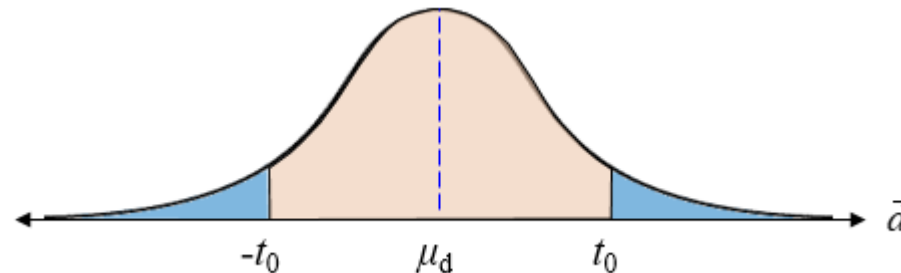
$$s_d = \sqrt{\frac{\sum (d - \bar{d})^2}{n - 1}} = \sqrt{\frac{\sum d^2 - \frac{(\sum d)^2}{n}}{n - 1}}$$

Testing the Difference Between Means (Dependent Samples)

Three conditions are required to conduct the test.

1. The samples must be randomly selected.
2. The samples must be dependent (paired).
3. Both populations must be normally distributed.

If these conditions are met, then the sampling distribution for $\bar{d}$ is approximated by a t -distribution with $n - 1$ degrees of freedom, where n is the number of data pairs.



Testing the Difference Between Means (Dependent Samples)

- The **test statistic** is $\bar{d} = \frac{\sum d}{n}$

- The **standardized test statistic** is

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

- The **degrees of freedom** are

$$\text{d.f.} = n - 1$$

First Case:

$$H_0: \mu_d =$$

$$H_a: \mu_d \neq$$

Second Case:

$$H_0: \mu_d \leq$$

$$H_a: \mu_d >$$

Third Case:

$$H_0: \mu_d \geq$$

$$H_a: \mu_d <$$



Example: Testing the Difference Between Means (Dependent Samples)

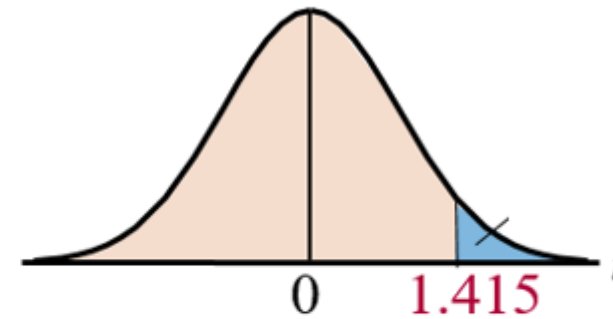
A golf club manufacturer claims that old scores are greater than new scores even if using the manufacturer's newly designed golf clubs. Eight golfers are randomly selected, and each is asked to give his or her most recent score. After using the new clubs for one month, the golfers are again asked to give their most recent score. The scores for each golfer are shown in the table. Assuming the golf scores are normally distributed, is there enough evidence to support the manufacturer's claim at $\alpha = 0.10$?

Golfer	1	2	3	4	5	6	7	8
Score (old)	89	84	96	82	74	92	85	91
Score (new)	83	83	92	84	76	91	80	91

Solution: Testing the Difference Between Means (Dependent Samples)

- $H_0: \mu_d \leq 0$
- $H_a: \mu_d > 0$
- $\alpha = 0.10$
- $\text{d.f.} = 8 - 1 = 7$
- critical value = 1.415
- Test Statistic: ?

- Rejection Region: ?



- Decision: ?

Solution: Obtaining test statistic

$d = (\text{old score}) - (\text{new score})$

Old	New	d	d^2
89	83	6	36
84	83	1	1
96	92	4	16
82	84	-2	4
74	76	-2	4
92	91	1	1
85	80	5	25
91	91	0	0
		$\Sigma = 13$	$\Sigma = 87$

$$\bar{d} = \frac{\Sigma d}{n} = \frac{13}{8} = 1.625$$

$$s_d = \sqrt{\frac{\Sigma d^2 - \frac{(\Sigma d)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{87 - \frac{(13)^2}{8}}{8-1}}$$

$$\approx 3.0677$$

- Test Statistic:**

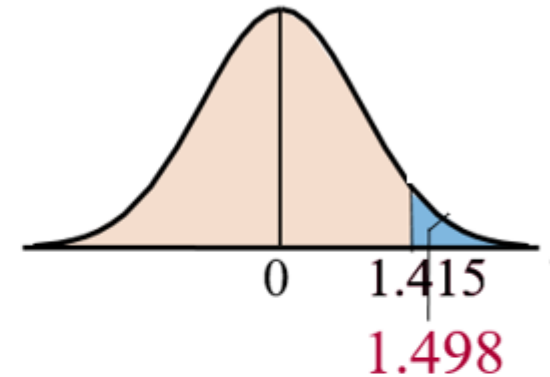
$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \approx \frac{1.625 - 0}{3.0677 / \sqrt{8}} \approx 1.498$$

Solution: Testing the Difference Between Means (Dependent Samples)

- $H_0: \mu_d \leq 0$
- $H_a: \mu_d > 0$
- $\alpha = 0.10$
- $\text{d.f.} = 8 - 1 = 7$
- critical value = 1.415
- Test Statistic:

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \approx \frac{1.625 - 0}{3.0677 / \sqrt{8}} \approx 1.498$$

- Rejection Region:



- Decision: **Reject H_0**
At the 10% level of significance, the results of this test indicate that after the golfers used the new clubs, their scores were significantly lower.

Used to test the difference between two population proportions, p_1 and p_2 .

Three conditions are required to conduct the test:

1. The samples must be randomly selected.
2. The samples must be independent.
3. The samples must be large enough to use a normal sampling distribution. That is,
 $n_1p_1 \geq 5$, $n_1q_1 \geq 5$, $n_2p_2 \geq 5$, and $n_2q_2 \geq 5$.

- If these conditions are met, then the sampling distribution for $\hat{p}_1 - \hat{p}_2$ is a normal distribution
- Mean: $\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$
- A weighted estimate of p_1 and p_2 can be found by using $\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}$, where $x_1 = n_1 \hat{p}_1$ and $x_2 = n_2 \hat{p}_2$
- Standard error:

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\bar{p}\bar{q} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}$$

- The **test statistic** is $\hat{p}_1 - \hat{p}_2$
- The **standardized test statistic** is

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} \text{ and } \bar{q} = 1 - \bar{p}$$

Note: $n_1\bar{p}$, $n_1\bar{q}$, $n_2\bar{p}$, and $n_2\bar{q}$ must be at least 5.

In a study of 200 randomly selected adult female and 250 randomly selected adult male Internet users, 30% of the females and 38% of the males said that they plan to shop online at least once during the next month. At $\alpha = 0.10$ test the claim that there is a difference between the proportion of female and the proportion of male Internet users who plan to shop online.

Solution:

1 = Females

2 = Males

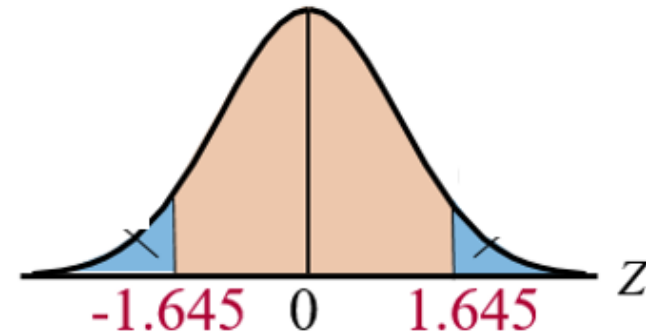
Solution: Testing the Difference Between Proportions

- $H_0: p_1 = p_2$
- $H_a: p_1 \neq p_2$
- $\alpha = 0.10$
- $n_1 = 200, n_2 = 250$

- critical value = -1.645 and 1.645

- Test Statistic:

- Rejection Region:



- Decision:

Solution: Obtaining test statistic

$$x_1 = n_1 \hat{p}_1 = 60 \quad x_2 = n_2 \hat{p}_2 = 95$$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{60 + 95}{200 + 250} \approx 0.3444$$

$$\bar{q} = 1 - \bar{p} = 1 - 0.3444 = 0.6556$$

Note:

$$n_1 \bar{p} = 200(0.3444) \geq 5 \quad n_1 \bar{q} = 200(0.6556) \geq 5$$

$$n_2 \bar{p} = 250(0.3444) \geq 5 \quad n_2 \bar{q} = 250(0.6556) \geq 5$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \approx \frac{(0.30 - 0.38) - (0)}{\sqrt{(0.3444) \cdot (0.6556) \cdot \left(\frac{1}{200} + \frac{1}{250}\right)}} \\ \approx -1.77$$

Solution: Obtaining test statistic

$$x_1 = n_1 \hat{p}_1 = 60 \quad x_2 = n_2 \hat{p}_2 = 95$$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{60 + 95}{200 + 250} \approx 0.3444$$

$$\bar{q} = 1 - \bar{p} = 1 - 0.3444 = 0.6556$$

Note:

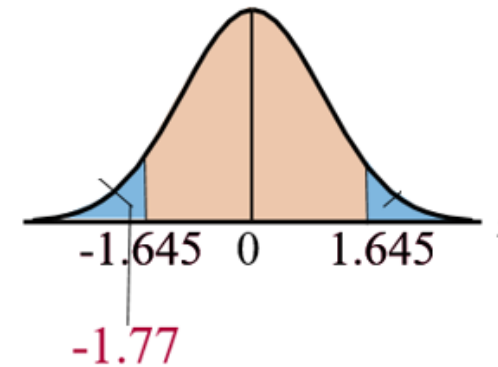
$$n_1 \bar{p} = 200(0.3444) \geq 5 \quad n_1 \bar{q} = 200(0.6556) \geq 5$$

$$n_2 \bar{p} = 250(0.3444) \geq 5 \quad n_2 \bar{q} = 250(0.6556) \geq 5$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \approx \frac{(0.30 - 0.38) - (0)}{\sqrt{(0.3444) \cdot (0.6556) \cdot \left(\frac{1}{200} + \frac{1}{250} \right)}} \\ \approx -1.77$$

- $H_0: p_1 = p_2$
- $H_a: p_1 \neq p_2$
- $\alpha = 0.10$
- $n_1 = 200, n_2 = 250$
- critical value = -1.645 and 1.645
- **Test Statistic:**
 $z = -1.77$

- **Rejection Region:**



- **Decision: Reject H_0**
At the 10% level of significance, there is enough evidence to conclude that there is a difference between the proportion of female and the proportion of male Internet users who plan to shop online.

Example: Testing the Difference Between Proportions

A medical research team conducted a study to test the effect of a cholesterol reducing medication. At the end of the study, the researchers found that of the 4700 randomly selected subjects who took the medication, 301 died of heart disease. Of the 4300 randomly selected subjects who took a placebo, 357 died of heart disease. At $\alpha = 0.01$ can you conclude that the death rate due to heart disease is lower for those who took the medication than for those who took the placebo? (*Adapted from New England Journal of Medicine*)

Solution:

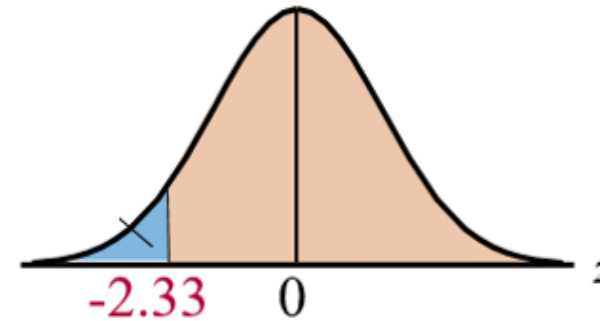
1 = Medication

2 = Placebo

Solution: Testing the Difference Between Proportions

- $H_0: p_1 \geq p_2$
- $H_a: p_1 < p_2$
- $\alpha = 0.01$
- $n_1 = 4700, n_2 = 4300$
- critical value = -2.33

- Rejection Region:



- Decision:

- Test Statistic:

Solution: Obtaining test statistic

$$\hat{p}_1 = \frac{x_1}{n_1} = \frac{301}{4700} = 0.064 \quad \hat{p}_2 = \frac{x_2}{n_2} = \frac{357}{4300} = 0.083$$

$$\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{301 + 357}{4700 + 4300} \approx 0.0731$$

$$\bar{q} = 1 - \bar{p} = 1 - 0.0731 = 0.9269$$

Note:

$$n_1\bar{p} = 4700(0.0731) \geq 5 \quad n_1\bar{q} = 4700(0.9269) \geq 5$$

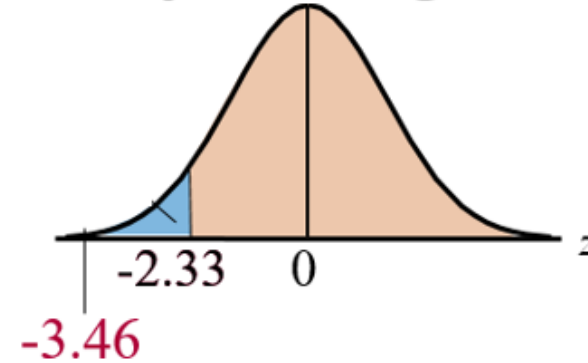
$$n_2\bar{p} = 4300(0.0731) \geq 5 \quad n_2\bar{q} = 4300(0.9269) \geq 5$$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\bar{p}\bar{q} \cdot \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} \approx \frac{(0.064 - 0.083) - (0)}{\sqrt{(0.0731) \cdot (0.9269) \cdot \left(\frac{1}{4700} + \frac{1}{4300} \right)}} \\ \approx -3.46$$

Solution: Testing the Difference Between Proportions

- $H_0: p_1 \geq p_2$
- $H_a: p_1 < p_2$
- $\alpha = 0.01$
- $n_1 = 4700, n_2 = 4300$
- critical value = -2.33
- **Test Statistic:**
 $z = -3.46$

- **Rejection Region:**



- **Decision: Reject H_0**
At the 1% level of significance, there is enough evidence to conclude that the death rate due to heart disease is lower for those who took the medication than for those who took the placebo.

8.1 Testing the Difference Between Means (Large Independent Samples)

8.2 Testing the Difference Between Means (Small Independent Samples)

8.3 Testing the Difference Between Means (Dependent Samples)

8.4 Testing the Difference Between Proportions