

Mathematics

Lecture 5

MAT 112
Additional Differentiation Topics

Chapter Objectives

- To develop a differentiation formula for $y = \ln u$.
 - To develop a differentiation formula for $y = e^u$.
 - To give a mathematical analysis of the economic concept of elasticity.
 - To discuss the notion of a function defined implicitly.
 - To show how to differentiate a function of the form u^v .
 - To approximate real roots of an equation by using calculus.
 - To find higher-order derivatives both directly and implicitly.
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- 1) Derivatives of Logarithmic Functions
 - 2) Derivatives of Exponential Functions
 - 3) Logarithmic Differentiation
 - 4) Higher-Order Derivatives
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1 Derivatives of Logarithmic Functions

- The derivatives of log functions are:

$$\text{a. } \frac{d}{dx}(\ln x) = \frac{1}{x} \ln \left(\lim_{h \rightarrow 0} \left(1 + \frac{h}{x} \right)^{x/h} \right)$$

$$\text{b. } \frac{d}{dx}(\ln|x|) = \frac{1}{x} \text{ where } x \neq 0$$

$$\text{c. } \frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx} \text{ for } u \neq 0$$

Example – Differentiating Functions Involving $\ln x$

a. Differentiate $f(x) = 5 \ln x$.

Solution: $f'(x) = 5 \frac{d}{dx}(\ln x) = \frac{5}{x}$ for $x > 0$

b. Differentiate $y = \frac{\ln x}{x^2}$

Solution:

$$y' = \frac{x^2 \frac{d}{dx}(\ln x) - (\ln x) \frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2 \left(\frac{1}{x} \right) - (\ln x)(2x)}{x^4}$$

$$= \frac{1 - 2\ln x}{x^3} \text{ for } x > 0$$

Example – Rewriting Logarithmic Functions before Differentiating

a. Find dy/dx if $y = \ln(2x+5)^3$.

Solution:

$$\frac{dy}{dx} = 3 \left(\frac{1}{2x+5} \right) (2)$$

$$= \frac{6}{2x+5} \text{ for } x > -5/2$$

b. Find $f'(p)$ if $f(p) = \ln((p+1)^2(p+2)^3(p+3)^4)$

Solution:

$$\begin{aligned} f'(p) &= 2\left(\frac{1}{p+1}\right)(1) + 3\left(\frac{1}{p+2}\right)(1) + 4\left(\frac{1}{p+3}\right)(1) \\ &= \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3} \end{aligned}$$

Procedure to Differentiate $\log_b u$

- Convert $\log_b u$ to $\frac{\ln u}{\ln b}$ and then differentiate.
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Example – Differentiating a Logarithmic Function to the Base 2

Differentiate $y = \log_2 x$.

Solution:
$$\frac{dy}{dx}(\log_2 x) = \frac{d}{dx} \left(\frac{\ln x}{\ln 2} \right) = \frac{1}{(\ln 2)x}$$

2 Derivatives of Exponential Functions

- The derivatives of exponential functions are:

a. $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

b. $\frac{d}{dx}(e^x) = e^x$

c. $\frac{d}{dx}(b^u) = b^u (\ln b) \frac{du}{dx}$

d. $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} \text{ for } f'(f^{-1}(x)) \neq 0$

e. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Example – Differentiating Functions Involving e^x

a. Find $\frac{d}{dx}(3e^x)$.

Solution: $\frac{d}{dx}(3e^x) = 3 \frac{d}{dx}(e^x) = 3e^x$

b. If $y = \frac{x}{e^x}$, find $\frac{dy}{dx}$.

Solution: $\frac{dy}{dx} = e^{-x} \frac{d}{dx} x + x \frac{d}{dx} (e^{-x}) = \frac{1-x}{e^x}$

c. Find y' when $y = e^2 + e^x + \ln 3$

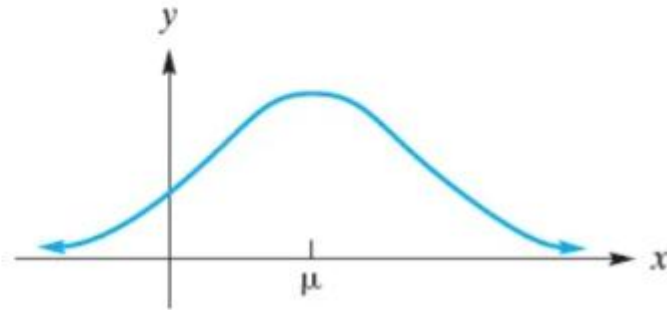
Solution: $y' = 0 + e^x + 0 = e^x$

Example – The Normal-Distribution Density Function

Determine the rate of change of y with respect to x

when $x = \mu + \sigma$.

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Example – Differentiating Different Forms

Find $\frac{d}{dx}(e^2 + x^e + 2^{\sqrt{x}})$

Solution:

$$\begin{aligned}\frac{d}{dx}(e^2 + x^e + 2^{\sqrt{x}}) &= ex^{e-1} + [e^{(\ln 2)\sqrt{x}}](\ln 2)\left(\frac{1}{2\sqrt{x}}\right) \\ &= ex^{e-1} + \frac{2^{\sqrt{x}} \ln 2}{2\sqrt{x}}\end{aligned}$$

Higher-Order Derivatives

For higher-order derivatives:

First derivative:	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}(f(x))$	$D_x y$
Second derivative:	y''	$f''(x)$	$\frac{d^2 y}{dx^2}$	$\frac{d^2}{dx^2}(f(x))$	$D_x^2 y$
Third derivative:	y'''	$f'''(x)$	$\frac{d^3 y}{dx^3}$	$\frac{d^3}{dx^3}(f(x))$	$D_x^3 y$
Fourth derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4 y}{dx^4}$	$\frac{d^4}{dx^4}(f(x))$	$D_x^4 y$

Example 1 – Finding Higher-Order Derivatives

a. If $f(x) = 6x^3 - 12x^2 + 6x - 2$, find all higher-order derivatives.

Solution:

$$f'(x) = 18x^2 - 24x + 6$$

$$f''(x) = 36x - 24$$

$$f'''(x) = 36$$

$$f^{(4)}(x) = 0$$

b. If $f(x) = 7$, find $f(x)$.

Solution:

$$f'(x) = 0$$

$$f''(x) = 0$$

$$f(x) = 3 \ln(x + 2) + 4 \ln(x + 1) + 2 \ln(x - 2)$$

$$f'(x) = \frac{3}{x+2} + \frac{4}{x+1} + \frac{2}{x-2}$$

$$f(x) = 2x^2 \ln x$$

$$f'(x) = (4x) (\ln x) + \left(\frac{1}{x}\right) 2x^2$$

Differentiate the following functions:

$$f(x) = e^{ax^2}$$

$$f'(x) = e^{ax^2} (2ax)$$

Differentiate the following functions:

$$f(x) = e^{x^2-2x+1}$$

$$f'(x) = e^{x^2-2x+1}[2x-2] = (2x-2)e^{x^2-2x+1}$$

$$f(x) = \ln x^2 = 2 \ln x$$

$$f'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$$

$$y = -x - x^2$$

$$\frac{dy}{dx} = -1 - 2x$$

$$y = x^3 + e^x$$

$$y' = 3x^2 + e^x$$

$$y = 8 - x$$

$$\frac{dy}{dx} = -1$$

$$f(x) = 3x^2 + 2x$$

$$f'(x) = 6x + 2$$

$$f(x) = \frac{1}{\sqrt{x}}$$

$$f'(x) = \frac{-1}{2} x^{-\frac{3}{2}}$$

$$f(x) = \left(\frac{6}{7}\right)^{\frac{2}{3}}$$

$$f'(x) = 0$$

$$f(x) = \frac{3}{5}x^6$$

$$f'(x) = \frac{18}{5}x^5$$

$$y = \frac{x+1}{x-1} \rightarrow y''$$

$$\begin{aligned} y' &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} \\ &= -\frac{2}{(x-1)^2} = -2(x-1)^{-2} \end{aligned}$$

$$\begin{aligned} y'' &= 4(x-1)^{-3} \\ &= \frac{4}{(x-1)^3} \end{aligned}$$

$$f(x) = \sqrt{x} \rightarrow f''(x)$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{\frac{3}{2}}}$$

Find $f'''(x)$:

If $f(x) = 3x^2 + 2x$

Solution

$$f'(x) = 6x + 2$$

$$f''(x) = 6$$

$$f'''(x) = \text{zero}$$

$$y = x^5 + x^4 + x^3 + x^2 + x + 1 \rightarrow y'''$$

$$y' = 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

$$y'' = 20x^3 + 12x^2 + 6x + 2$$

$$y''' = 60x^2 + 24x + 6$$

$$y = 4x^3 - 12x^2 + 6x + 2 \rightarrow y'''$$

$$y' = 12x^2 - 24x + 6$$

$$y'' = 24x - 24$$

$$y''' = 24$$

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