

Mathematics

Lecture 4

MAT 122

Operations of Matrices

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Matrix Addition

Matrix Subtraction

Matrix Multiplication

Practice

Sum $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of **A** and **B**.

Matrix Addition

Consider a snowmobile dealer who sells two models, Deluxe and Super. Each is available in one of two colors, red and blue. Suppose that the sales for January and February are represented by the matrices

$$J = \begin{matrix} & \text{Deluxe} & \text{Super} \\ \text{red} & 1 & 2 \\ \text{blue} & 3 & 5 \end{matrix} \quad F = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

respectively. Each row of J and F gives the number of each model sold for a given color. Each column gives the number of each color sold for a given model. A matrix representing total sales for each model and color over the two months can be obtained by adding the corresponding entries in J and F :

$$\begin{bmatrix} 4 & 3 \\ 7 & 7 \end{bmatrix}$$

Definition

If A and B are both $m \times n$ matrices, then the *sum* $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B ; so that $(A + B)_{ij} = A_{ij} + B_{ij}$. If the size of A is different from the size of B , then $A + B$ is not defined.

For example, let

$$A = \begin{bmatrix} 3 & 0 & -2 \\ 2 & -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & -3 & 6 \\ 1 & 2 & -5 \end{bmatrix}$$

Since A and B are the same size (2×3), their sum is defined. We have

$$A + B = \begin{bmatrix} 3+5 & 0+(-3) & -2+6 \\ 2+1 & -1+2 & 4+(-5) \end{bmatrix} = \begin{bmatrix} 8 & -3 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

EXAMPLE 1 Matrix Addition

a.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ -6 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1+7 & 2-2 \\ 3-6 & 4+4 \\ 5+3 & 6+0 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -3 & 8 \\ 8 & 6 \end{bmatrix}$$

b. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not defined, since the matrices are not the same size.

If A , B , C , and O have the same size, then the following properties hold for matrix addition:

Properties of Matrix Addition

- | | |
|--------------------------------|----------------------|
| 1. $A + B = B + A$ | commutative property |
| 2. $A + (B + C) = (A + B) + C$ | associative property |
| 3. $A + O = A = O + A$ | identity property |

Property 1 states that matrices can be added in any order, and Property 2 allows matrices to be grouped for the addition operation. Property 3 states that the zero matrix plays the same role in matrix addition as does the number 0 in the addition of real numbers. These properties are illustrated in the next example.

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- a. Show that $A + B = B + A$.
- b. Show that $A + (B + C) = (A + B) + C$.
- c. Show that $A + O = A$.

Solution:

$$A + B = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & 2 \end{bmatrix} \quad B + A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & 2 \end{bmatrix}$$

Thus, $A + B = B + A$.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. Show that $A + (B + C) = (A + B) + C$.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + (B + C) = A + \begin{bmatrix} -2 & 2 & 1 \\ 1 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 4 & 2 \\ -1 & -5 & 3 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -3 & 2 \end{bmatrix} + C = \begin{bmatrix} -1 & 4 & 2 \\ -1 & -5 & 3 \end{bmatrix}$$

c. Show that $A + O = A$.

Solution:

$$A + O = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{bmatrix} = A$$

Definition of Subtraction of Matrices

Mathematically, if there are two matrices, say $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same order, say $m \times n$, then the subtraction of A and B, i.e., $A - B$ is defined as:

Matrix $D = [d_{ij}]$

$$A - B = a_{ij} - b_{ij}$$

Thus,

$$d_{ij} = a_{ij} - b_{ij}, (i = 1, 2, 3, \dots \text{ and } j = 1, 2, 3, \dots)$$

Subtraction of 2 x 2 Matrices

Suppose A and B are 2 x 2 matrices, such that;

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then, subtraction of matrices A and B, will be given as:

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}$$

Fact: If A and B are two matrices of the same order, then;

$$A - B \neq B - A$$

Thus, commutative law is not applicable for subtraction of matrices.

Subtraction of 3 x 3 Matrices

Suppose A and B are 3 x 3 matrices, such that;

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Then, subtraction of matrices A and B, will be given as:

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & a_{13} - b_{13} \\ a_{21} - b_{21} & a_{22} - b_{22} & a_{23} - b_{23} \\ a_{31} - b_{31} & a_{32} - b_{32} & a_{33} - b_{33} \end{bmatrix}$$

If A and B are two matrices. Then find subtraction of matrices A and B.

$$A = \begin{pmatrix} 3 & 5 \\ 9 & 8 \end{pmatrix}$$

And

$$B = \begin{pmatrix} 1 & 3 \\ 8 & 9 \end{pmatrix}$$

Solution:

$$A-B = \begin{pmatrix} 3 & 5 \\ 9 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 8 & 9 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 3-1 & 5-3 \\ 9-8 & 8-9 \end{pmatrix}$$

$$A-B = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}$$

Besides the operations of matrix addition and scalar multiplication, the product AB of matrices A and B can be defined under a certain condition, namely, that *the number of columns of A is equal to the number of rows of B* . Although the following definition

Definition

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then the product AB is the $m \times p$ matrix with entry $(AB)_{ik}$ given by

$$(AB)_{ik} = \sum_{j=1}^n A_{ij}B_{jk} = A_{i1}B_{1k} + A_{i2}B_{2k} + \cdots + A_{in}B_{nk}$$

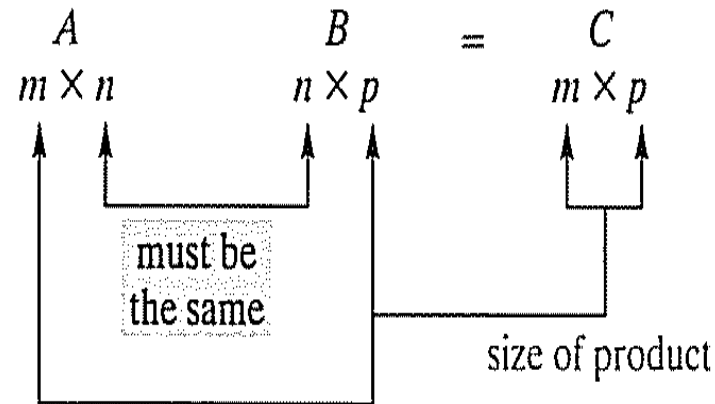
In words, $(AB)_{ik}$ is obtained by summing the products formed by multiplying, in order, each entry in row i of A by the corresponding entry in column k of B . If the number of columns of A is not equal to the number of rows of B , then the product AB is not defined.

Observe that the definition applies when A is a row vector with n entries and B is a column vector with n entries. In this case A is $1 \times n$, B is $n \times 1$, and AB is 1×1 . (We noted in Section 6.1 that a 1×1 matrix is just a *number*.) In fact,

$$\text{if } A = [A_1 \quad A_2 \quad \cdots \quad A_n] \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

$$\text{then } AB = \sum_{j=1}^n A_j B_j = A_1 B_1 + A_2 B_2 + \cdots + A_n B_n$$

Three points must be completely understood concerning this definition of AB . First, the number of columns of A must be equal to the number of rows of B . Second, the product AB has as many rows as A and as many columns as B .



Find the multiplication of AB

$$A = \begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

Thus,

$$AB = \begin{bmatrix} 2 & 1 & -6 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & 2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14 & -2 & -10 \\ -3 & -10 & -7 \end{bmatrix}$$

Compute the matrix product

$$AB = \begin{bmatrix} 2 & -4 & 2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -10 \\ -6 & -2 \end{bmatrix}$$

Compute AB and BA if

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}.$$

Compute AB and BA if

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix}.$$

Solution: We have

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -5 & 7 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 14 & 3 \end{bmatrix}$$

Note that although both AB and BA are defined, and the same size, AB and BA are not equal.

Let's Practice

$$\text{If } A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Find a) $A + B$

b) $B + C$

c) $A + C$

f) $A - C$

d) $A - B$

e) $B - C$

h) $A + D$

$$\begin{aligned} A+B &= \begin{bmatrix} (2+1) & (-3+(-2)) & (4+0) \\ (0+5) & (2+1) & (1+2) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -5 & 4 \\ 5 & 3 & 3 \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} B+C &= \begin{bmatrix} (1+(-3)) & (-2+0) & (0+5) \\ (5+2) & (1+1) & (2+3) \end{bmatrix} \\ &= \begin{bmatrix} -2 & -2 & 5 \\ 7 & 2 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A + C &= \begin{bmatrix} (2 + (-3)) & (-3 + 0) & (4 + 5) \\ (0 + 2) & (2 + 1) & (1 + 3) \end{bmatrix} \\ &= \begin{bmatrix} -1 & -3 & 9 \\ 2 & 3 & 4 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A + -B &= \begin{bmatrix} (2 - 1) & (-3 - (-2)) & (4 - 0) \\ (0 + 2) & (2 - 1) & (1 - 2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 4 \\ -5 & 1 & -1 \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$\begin{aligned} B - C &= \begin{bmatrix} (1 - (-3)) & (-2 - 0) & (0 - 5) \\ (5 - 2) & (1 - 1) & (2 - 3) \end{bmatrix} \\ &= \begin{bmatrix} 4 & -2 & -5 \\ 3 & 0 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A - C &= \begin{bmatrix} (2 - (-3)) & (-3 - 0) & (4 - 5) \\ (0 - 2) & (2 - 1) & (1 - 3) \end{bmatrix} \\ &= \begin{bmatrix} 5 & -3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \end{aligned}$$

$A + D$

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -3 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

It is impossible to find the value of this addition because it should be in the same size

If

$$A = \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 4 \\ 7 & -3 \end{bmatrix}$$

Find AB and BA

$$AB = \begin{bmatrix} (3 \times -1) + (5 \times 7) & (3 \times 4) + (5 \times -3) \\ (10 \times -1) + (0 \times 7) & (10 \times 4) + (0 \times -3) \end{bmatrix} = \begin{bmatrix} 32 & -3 \\ -10 & 40 \end{bmatrix}$$

$$BA = \begin{bmatrix} (-1 \times 3) + (4 \times 10) & (-1 \times 5) + (4 \times 0) \\ (7 \times 3) + (-3 \times 10) & (7 \times 5) + (-3 \times 0) \end{bmatrix} = \begin{bmatrix} 37 & -5 \\ -9 & 35 \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix}$

Find:

a) $2A$

b) $-A$

c) A^2

$$\text{a) } 2A = 2 \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} (3 \times 2) & (5 \times 2) \\ (10 \times 2) & (0 \times 2) \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 20 & 0 \end{bmatrix}$$

$$\text{b) } -A = -1 \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ -10 & 0 \end{bmatrix}$$

$$\text{c) } A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 10 & 0 \end{bmatrix} = \begin{bmatrix} 59 & 15 \\ 30 & 50 \end{bmatrix}$$

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- **Matrix Addition**
- **Matrix Subtraction**
- **Matrix Multiplication**