

Business Mathematics

Prepared by Dr. Alfaisal Hasan

MTH 112
Business Mathematics

Lecture NO 10



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Chapter

Topics

- Chapter 1: Mathematical preliminaries
- Chapter 2: The straight line and applications
- Chapter 3: Functions and Graphs
- Chapter 4: Limits and Continuity
- Chapter 5: Financial Mathematics
- Chapter 6: Differentiation
- Chapter 7: Methods of Differentiation
- Chapter 8: Function of several variables
- Chapter 9: Integration and applications
- **Chapter 10: Linear algebra and applications**

Matrices Notation and Terminology

- A general $m \times n$ matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The entry that occurs in row i and column j of matrix A will be denoted a_{ij} .

Linear Algebra and Applications

- A matrix A with n rows and n columns is called a **square matrix of order n** , and the shaded entries are said to be on the **main diagonal** of A .

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Linear Algebra and Applications

Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size,
then $A = B$ if and only if $a_{ij} = b_{ij}$ for all i and j .

Linear Algebra and Applications

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

- If $x=5$, then $A=B$.
- For all other values of x , the matrices A and B are not equal.
- There is no value of x for which $A=C$ since A and C have different sizes.

Linear Algebra and Applications

- Rectangular display of vectors in rows and columns
- Can inform about the same vector intensity at different times or different voxels at the same time
- Vector is just a $n \times 1$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix}$$

Square (3 x 3)

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 8 \end{bmatrix}$$

Rectangular (3 x 2)

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

d_{ij} : i^{th} row, j^{th} column

Defined as rows x columns (R x C)

Linear Algebra and Applications

If A is any matrix and c is any scalar, then the **product** cA is the matrix obtained by multiplying each entry of the matrix A by c . The matrix cA is said to be the **scalar multiple** of A .

In matrix notation, if $A = [a_{ij}]$, then

$$(cA)_{ij} = ca_{ij}$$

Linear Algebra and Applications

Chapter

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix}$$

$$2\mathbf{A} = \begin{bmatrix} 2 & 4 & 6 \\ 10 & 8 & 2 \\ 12 & 14 & 8 \end{bmatrix}$$

Linear Algebra and Applications

$$\mathbf{C} = \begin{bmatrix} 1 & 4 \\ 0 & 7 \\ 3 & 8 \end{bmatrix}$$

$$-3\mathbf{C} = \begin{bmatrix} -3 & -12 \\ 0 & -21 \\ -9 & -24 \end{bmatrix}$$

Linear Algebra and Applications

- For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

- We have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

- It common practice to denote $(-1)B$ by $-B$.

Linear Algebra and Applications

- If A and B are matrices of the same size, then the **sum** $A+B$ is the matrix obtained by adding the entries of B to the corresponding entries of A .
- Vice versa, the **difference** $A-B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A .
- Note: Matrices of different sizes cannot be added or subtracted.

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}$$

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Linear Algebra and Applications

- Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix}, \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

- The expressions $A+C$, $B+C$, $A-C$, and $B-C$ are undefined.

Linear Algebra and Applications

Addition

- Commutative: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- Associative: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2+1 & 4+0 \\ 2+3 & 5+1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Linear Algebra and Applications

Transposition

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{b}^T = [1 \quad 1 \quad 2]$$

$$\mathbf{d} = [3 \quad 4 \quad 9]$$

$$\mathbf{d}^T = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}$$

column



row

row



column

$$\mathbf{A} = \begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{3} \\ \boxed{5} & \boxed{4} & \boxed{1} \\ \boxed{6} & \boxed{7} & \boxed{4} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} \boxed{1} & \boxed{5} & \boxed{6} \\ \boxed{2} & \boxed{4} & \boxed{7} \\ \boxed{3} & \boxed{1} & \boxed{4} \end{bmatrix}$$

Linear Algebra and Applications

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 4 & 7 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\left(\mathbf{A}^T\right)^T = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 1 \\ 6 & 7 & 4 \end{bmatrix} = \mathbf{A}$$

Linear Algebra and Applications

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ 3 \ 5], \quad D = [4]$$
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Linear Algebra and Applications

- Observe that
- In the special case where A is a **square** matrix, the transpose of A can be obtained by interchanging entries that are symmetrically positioned about the main diagonal.

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Definition

- If A is a square matrix, then the **trace of A** , denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

Linear Algebra and Applications

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Linear Algebra and Applications

- Multiplication method:

Sum over product of respective rows and columns

$$\begin{array}{ccc}
 \begin{array}{c} \textcircled{\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}} \\ \mathbf{A} \end{array} & \begin{array}{c} \textcircled{\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}} \\ \mathbf{B} \end{array} & = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{array}{l} \text{Define output} \\ \text{matrix} \end{array} \\
 & & = \begin{bmatrix} (1 \times 2) + (0 \times 3) & (1 \times 1) + (0 \times 1) \\ (2 \times 2) + (3 \times 3) & (2 \times 1) + (3 \times 1) \end{bmatrix} \\
 & & = \begin{bmatrix} 2 & 1 \\ 13 & 5 \end{bmatrix}
 \end{array}$$

Linear Algebra and Applications

- Matrix multiplication is NOT commutative
- $AB \neq BA$
- Matrix multiplication IS associative
- $A(BC) = (AB)C$
- Matrix multiplication IS distributive
- $A(B+C) = AB+AC$
- $(A+B)C = AC+BC$

Identity matrix

Is there a matrix which plays a similar role as the number 1 in number multiplication?

Consider the $n \times n$ matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Algebra and Applications

Matrix inverse

- **Definition.** A matrix A is called **nonsingular** or **invertible** if there exists a matrix A^{-1} such that:

$$AA^{-1} = A^{-1}A = I_2$$

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Algebra and Applications

Matrix inverse $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = (ad - bc) \neq 0$$

Singular matrix if $\det(A) = 0$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Linear Algebra and Applications

Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Linear Algebra and Applications

Chapter

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

$$\det(A) = (3)(2) - (1)(5) = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

Linear Algebra and Applications

Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix}$$

Linear Algebra and Applications

$$A = \begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix}$$

$$\det(A) = (4)(3) - (-3)(2) = 12 + 6 = 18$$

$$A^{-1} = \frac{1}{18} \begin{bmatrix} 3 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{18} & \frac{3}{18} \\ \frac{-2}{18} & \frac{4}{18} \end{bmatrix}$$

Linear Algebra and Applications

Find the inverse of the matrix

$$B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

Linear Algebra and Applications

$$B = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$$

$$\det(B) = (4)(2) - (1)(8) = 0$$

$\det(B) = 0$, Matrix B is Singular matrix

Matrix B has no inverse

Linear Algebra and Applications

Determinants

If $A = [a_{ij}]$ is a square matrix of order 1,

then $|A| = |a_{11}| = a_{11}$

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example

Evaluate the determinant : $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$

Linear Algebra and Applications

Evaluate the determinant: $\begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix}$

$$\text{Solution : } \begin{vmatrix} 4 & -3 \\ 2 & 5 \end{vmatrix} = 4 \times 5 - 2 \times (-3) = 20 + 6 = 26$$

Linear Algebra and Applications

Solution of System of 2 Linear Equations (Cramer's Rule)

Let the system of linear equations be

$$a_1x + b_1y = c_1 \quad \dots(i)$$

$$a_2x + b_2y = c_2 \quad \dots(ii)$$

Then $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ provided $D \neq 0$,

$$\text{where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Linear Algebra and Applications

$$a_1x + b_1y = c_1 \quad \dots(i)$$

$$a_2x + b_2y = c_2 \quad \dots(ii)$$

$$\text{where } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Note :

If $D \neq 0$,

then the system is consistent and has unique solution.

Example

Using Cramer's rule , solve the following
system of equations $2x-3y=7$, $3x+y=5$

Linear Algebra and Applications

Using Cramer's rule , solve the following system of equations $2x-3y=7$, $3x+y=5$

$$D = \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 2 + 9 = 11 \neq 0$$

$$D_1 = \begin{vmatrix} 7 & -3 \\ 5 & 1 \end{vmatrix} = 7 + 15 = 22$$

$$D_2 = \begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix} = 10 - 21 = -11$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{22}{11} = 2 \text{ and } y = \frac{D_2}{D} = \frac{-11}{11} = -1$$

Example

Using Cramer's rule , solve the following
system of equations $x+2y=10$, $5x+8y=40$

Linear Algebra and Applications

Using Cramer's rule , solve the following
system of equations $x+2y=10$, $5x+8y=40$

$$D = \begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix} = 8 - 10 = -2 \neq 0$$

$$D_1 = \begin{vmatrix} 10 & 2 \\ 40 & 8 \end{vmatrix} = 80 - 80 = 0$$

$$D_2 = \begin{vmatrix} 1 & 10 \\ 5 & 40 \end{vmatrix} = 40 - 50 = -10$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{0}{-2} = 0 \text{ and } y = \frac{D_2}{D} = \frac{-10}{-2} = 5$$

Example

Using Cramer's rule , solve the following
system of equations $3x+y=12$, $5x-y=4$

Linear Algebra and Applications

Using Cramer's rule , solve the following
system of equations $3x+y=12$, $5x-y=4$

$$D = \begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = -3 - 5 = -8 \neq 0$$

$$D_1 = \begin{vmatrix} 12 & 1 \\ 4 & -1 \end{vmatrix} = -12 - 4 = -16$$

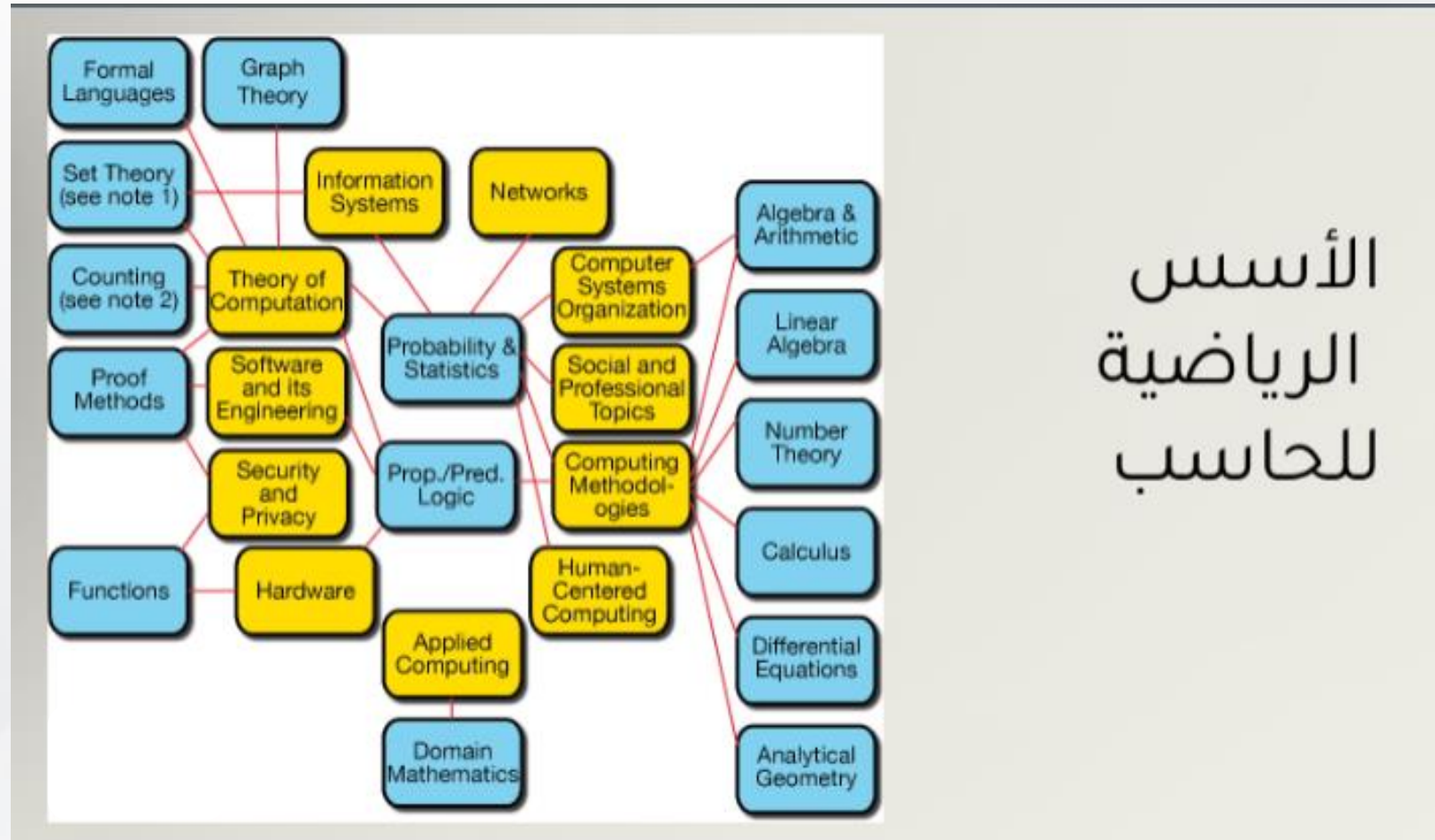
$$D_2 = \begin{vmatrix} 3 & 12 \\ 5 & 4 \end{vmatrix} = 12 - 60 = -48$$

$$\therefore D \neq 0$$

$$\therefore \text{By Cramer's Rule } x = \frac{D_1}{D} = \frac{-16}{-8} = 2 \text{ and } y = \frac{D_2}{D} = \frac{-48}{-8} = 6$$

AI in Mathematics

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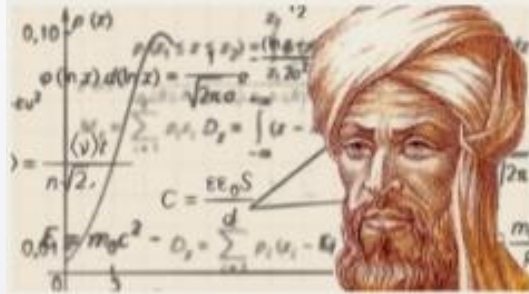


الأسس
الرياضية
للحاسب

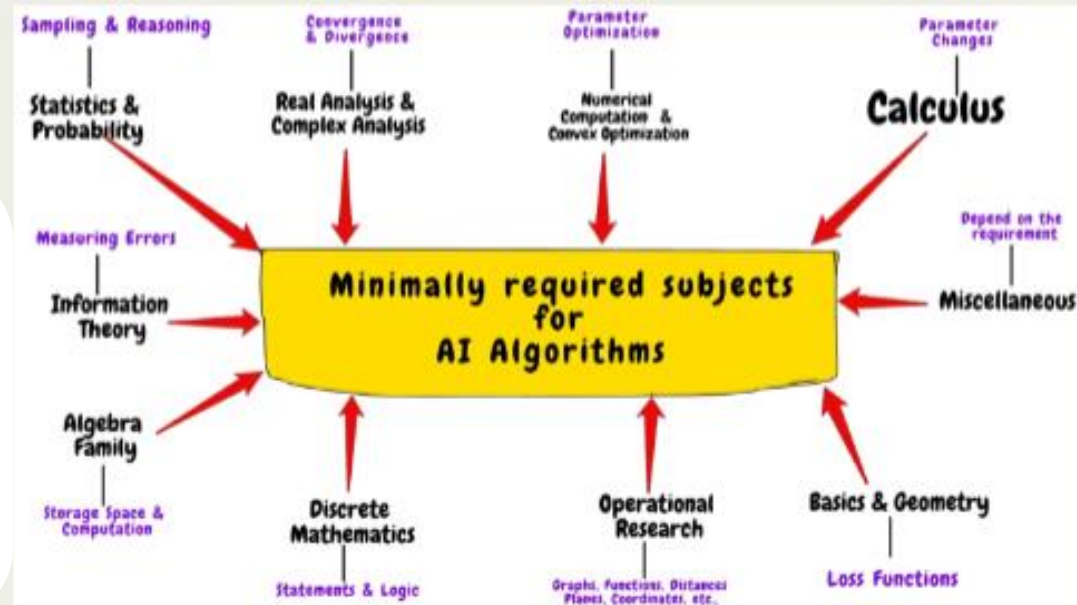
AI in Mathematics

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الخوارزميات

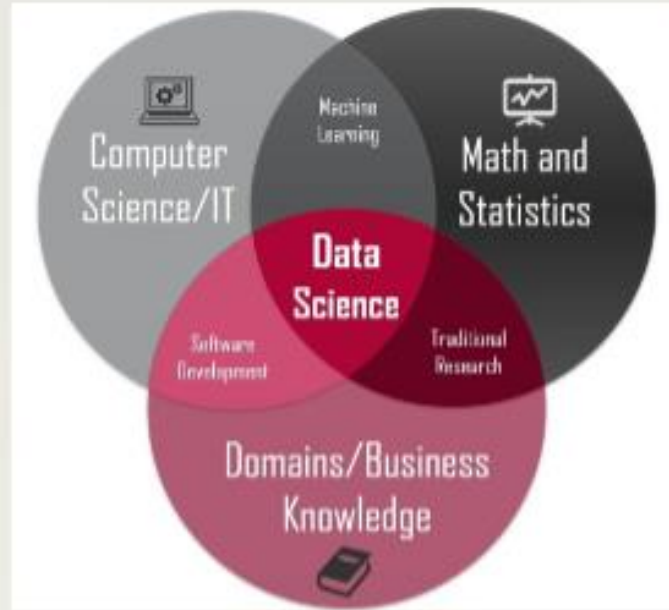


Algorithm



AI in Mathematics

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علم البيانات

Data Analysis تحليل البيانات

علم التشفير



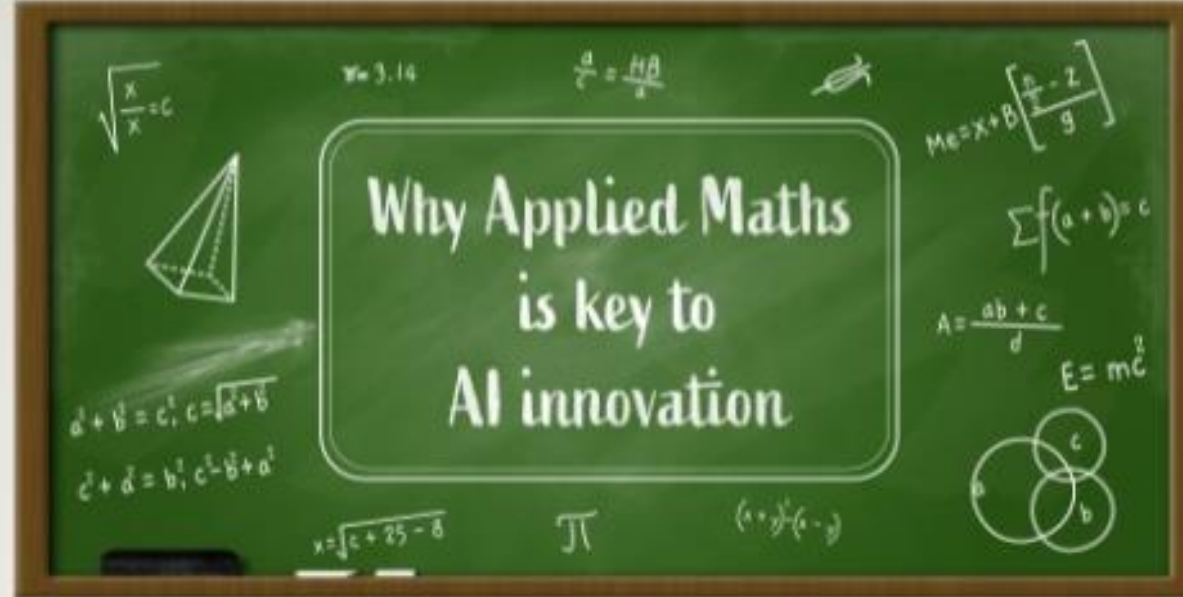
**On some Algebraic
methods in Cryptography**

يستخدم التشفير
مفاهيم رياضية
معقدة لضمان أمان
المعلومات
وحمايتها من
الوصول غير المصرح
به. تشمل
التطبيقات عملية
تشفير البيانات وفك
تشفيرها بواسطة
خوارزميات رياضية
معترف بها.

AI in Mathematics

المحاكاة

تعتبر الرياضيات التطبيقية
ضرورية لتطوير نماذج
رياضية تحاكي الواقع
وتساعد في حل المشكلات
العملية. تشمل تطبيقاتها
في الحوسبة تطوير
البرمجيات والنمذجة
الرياضية.



AI in Mathematics

الذكاء الاصطناعي
هو قدرة الأنظمة
البرمجية على محاكاة
وظائف التفكير
البشري مثل التعلم،
الاستدلال، واتخاذ
القرارات. يتم
استخدامه في
مجالات علمية
وحياتية متعددة

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تخصيص التعليم

يمكن للذكاء الاصطناعي
تخصيص مناهج الرياضيات
لتناسب مستوى كل طالب،
مما يساعد على تعزيز الفهم
واستيعاب المفاهيم.
التطبيقات الذكية تستخدم
لتحديد نقاط القوة والضعف
لدى الطلاب وتقديم محتوى
مخصص.



Un Solved Examples

Chapter

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

Find $3\mathbf{A}$ and $\text{Tr}(\mathbf{A})$ and $\mathbf{A}^T$

Un Solved Examples

Chapter

Find the inverse of the matrix

$$A = \begin{bmatrix} 5 & 1 \\ 5 & 2 \end{bmatrix}$$

Un Solved Examples

Chapter

Find the inverse of the matrix

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

Un Solved Examples

Chapter

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

Find $A + B$, $A - B$ and AB

Un Solved Examples

Chapter

Using Cramer's rule , solve the
following system of equations

$$2x+y=1, 4x-y=5$$



Thank You