

Mathematics

Lecture 3

MAT 112
Limits and Continuity

Prepared by
Dr. Israa Lewaa

- To study limits and their basic properties.
 - To study one-sided limits, infinite limits, and limits at infinity.
 - To study continuity and to find points of discontinuity for a function.
-

Limits

Continuity

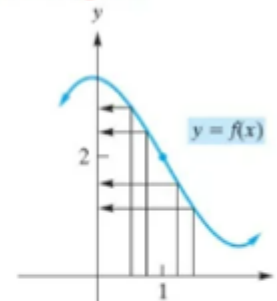
- The limit of $f(x)$ as x approaches a is the number L , written as

$$\lim_{x \rightarrow a} f(x) = L$$

Example 1 – Estimating a Limit from a Graph

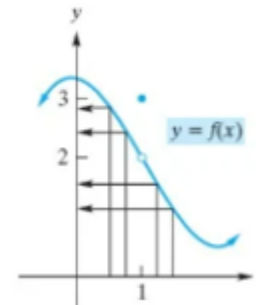
a. Estimate $\lim_{x \rightarrow 1} f(x)$ from the graph.

Solution: $\lim_{x \rightarrow 1} f(x) = 2$



b. Estimate $\lim_{x \rightarrow 1} f(x)$ from the graph.

Solution: $\lim_{x \rightarrow 1} f(x) = 2$



Properties of Limits

1. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c = c$ where c is a constant

2. $\lim_{x \rightarrow a} x^n = a^n$ for any positive integer n

3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

4. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

5. $\lim_{x \rightarrow a} [cf(x)] = c \cdot \lim_{x \rightarrow a} f(x)$

Properties of Limits

1. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} c = c$ where c is a constant

2. $\lim_{x \rightarrow a} x^n = a^n$ for any positive integer n

3. $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

4. $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

5. $\lim_{x \rightarrow a} [cf(x)] = c \cdot \lim_{x \rightarrow a} f(x)$

Properties of Limits

$$6. \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} \text{ if } \lim_{x \rightarrow a} g(x) \neq 0$$

$$7. \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

It's very important to substitute first by the value of x, If we have undefined quantity like:

- $\frac{0}{0}$
- $\infty - \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 1^∞
- $\infty \times 0$
- 0^0

Then, we have to use factorization.

Example – Applying Limit Properties 1 and 2

a. $\lim_{x \rightarrow 2} 7 = 7;$ $\lim_{x \rightarrow -5} 7 = 7$

b. $\lim_{x \rightarrow 6} x^2 = 6^2 = 36$

c. $\lim_{t \rightarrow -2} t^4 = (-2)^4 = 16$

Example – Limit of a Polynomial Function

Find an expression for the polynomial function,

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0$$

Solution:

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0)$$

$$= c_n \lim_{x \rightarrow a} x^n + c_{n-1} \lim_{x \rightarrow a} x^{n-1} + \dots + c_1 \lim_{x \rightarrow a} x + \lim_{x \rightarrow a} c_0$$

$$= c_n a^n + c_{n-1} a^{n-1} + \dots + c_1 a + c_0$$

$$= f(a)$$

where $\lim_{x \rightarrow a} f(x) = f(a)$

Limits at Infinity for Rational Functions

- If $f(x)$ is a rational function,

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{a_n x^n}{b_m x^m} \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{a_n x^n}{b_m x^m}$$

Example – Limits at Infinity for Polynomial Functions

Find the limit (if it exists)

a. $\lim_{x \rightarrow -\infty} (x^3 - x^2 + x - 2)$

b. $\lim_{x \rightarrow \infty} (-2x^3 + 9x)$

Solution: $\lim_{x \rightarrow -\infty} (x^3 - x^2 + x - 2) = -\infty$

Solution: $\lim_{x \rightarrow \infty} (-2x^3 + 9x) = \infty$

Example 1:

To Compute

$$\lim_{x \rightarrow -4} (5x^2 + 8x - 3)$$

Solution:

First, use property 2 to divide the limit into three separate limits. Then use property 1 to bring the constants out of the first two. This gives,

$$\begin{aligned}\lim_{x \rightarrow -4} (5x^2 + 8x - 3) &= \lim_{x \rightarrow -4} (5x^2) + \lim_{x \rightarrow -4} (8x) - \lim_{x \rightarrow -4} (3) \\ &= 5(-4)^2 + 8(-4) - 3 \\ &= 80 - 32 - 3 = 45\end{aligned}$$

Example 2:

To Compute

$$\lim_{x \rightarrow 6} \left[\frac{(x-3)(x-2)}{x-4} \right]$$

Solution:

Given

$$\lim_{x \rightarrow 6} \left[\frac{(x-3)(x-2)}{x-4} \right]$$

$$= \left[\frac{\lim_{x \rightarrow 6} (x-3) \lim_{x \rightarrow 6} (x-2)}{\lim_{x \rightarrow 6} (x-4)} \right]$$

$$= \left[\frac{(6-3)(6-2)}{(6-4)} \right]$$

$$= \left[\frac{(3)(4)}{(2)} \right] = 6$$

It's very important to **substitute** first by the value of x , If we have undefined quantity like:

- $\frac{0}{0}$
- $\infty - \infty$
- $\frac{\infty}{\infty}$
- ∞^0
- 1^∞
- $\infty \times 0$
- 0^0

Then, we have to use **Factorization.**

Example 3:

Compute

$$\lim_{x \rightarrow 3} \frac{(x^2 - 9)}{x - 3}$$

Solution: Given

$$\lim_{x \rightarrow 3} \frac{(x^2 - 9)}{x - 3}$$

It is to be noted that, on substituting the value 3 directly to the function, the numerator as well as denominator will become 0, and we know the value

$\frac{0}{0}$, does not exist.

Using the property of squares, we have:

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \rightarrow 3} (x + 3)$$

$$= 6$$

FACTORISATION


$$k(a + b) = k \times a + k \times b$$

Example:

$$3(x + 2y) = 3x + 6y$$

FACTORISATION


$$(a + b)(c + d) = a \times c + a \times d + b \times c + b \times d$$

Example:

$$(3+x)(x+2y) = 3x + 6y + x^2 + 2xy$$

Factorization by **Common factor**

$$x^2 + 3x = x(x + 3).$$

Example:

$$3x + 9 = 3(x+3).$$

Factorization for Quadratic Polynomial

$$x^2 + (a + b) x + ab = (x+a)(x + b).$$

Example:

$$x^2 + 5x + 6 = (x+2) (x+3)$$

Factorization for the difference between Two squares

$$a^2 - b^2 = (a - b)(a + b)$$

Example:

$$4x^2 - 9 = (2x + 3)(2x - 3)$$

Example:

$$\text{Find } \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

Solution:

We can't find the limit by substituting $x = 1$ because

$$\frac{(1)^2 - 1}{1 - 1} = \frac{0}{0} \text{ is undefined.}$$

Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares and then cancel out the common term $(x - 1)$

Therefore,

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 = 2 \end{aligned}$$

Evaluate $\lim_{x \rightarrow 2} (8 - 3x + 12x^2)$, if it exists.

Solution

$$\lim_{x \rightarrow 2} (8 - 3x + 12x^2)$$

$$= 8 - 3(2) + 12(4) = \boxed{50}$$

Example – Finding a Limit

Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}.$

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x + 1}$$

$$= \frac{(1)^2 - 1}{1 + 1} = \frac{1 - 1}{2} = 0$$

Solution: $\lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1}$

$$= \frac{(-1)^2 - 1}{-1 + 1} = \frac{1 - 1}{-1 + 1} = \frac{0}{0} \quad (\text{Undefined Quantity})$$

So, we will use the factorization Method

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^2 - 1}{x + 1} &= \frac{(x + 1)(x - 1)}{(x + 1)} \\ &= \lim_{x \rightarrow -1} (x - 1) = -1 - 1 = -2 \end{aligned}$$

Evaluate $\lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15}$, if it exists.

Solution:

$$\begin{aligned} & \lim_{x \rightarrow -5} \frac{x^2 - 25}{x^2 + 2x - 15} \\ &= \frac{(-5)^2 - 25}{(-5)^2 + 2(-5) - 15} = \frac{0}{0} \quad (\text{Undefined Quantity}) \end{aligned}$$

So, we need to use factorization to obtain the value of limit

$$= \lim_{x \rightarrow -5} \frac{(x - 5)(x + 5)}{(x - 3)(x + 5)}$$

$$= \lim_{x \rightarrow -5} \frac{x - 5}{x - 3} = \boxed{\frac{5}{4}}$$

Evaluate $\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1}$, if it exists.

Solution

$$\lim_{t \rightarrow -3} \frac{6 + 4t}{t^2 + 1} :$$

$$= \frac{-6}{10} = \boxed{-\frac{3}{5}}$$

Limits and Algebraic Manipulation

- If $f(x) = g(x)$ for all $x \neq a$, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

Definition

- $f(x)$ is **continuous** if three conditions are met:
 1. $f(x)$ exists
 2. $\lim_{x \rightarrow a} f(x)$ exists
 3. $\lim_{x \rightarrow a} f(x) = f(a)$
-

Example 1 – Applying the Definition of Continuity

- a. *Show that $f(x) = 5$ is continuous at 7.*
 - b. *Show that $g(x) = x^2 - 3$ is continuous at -4 .*
-

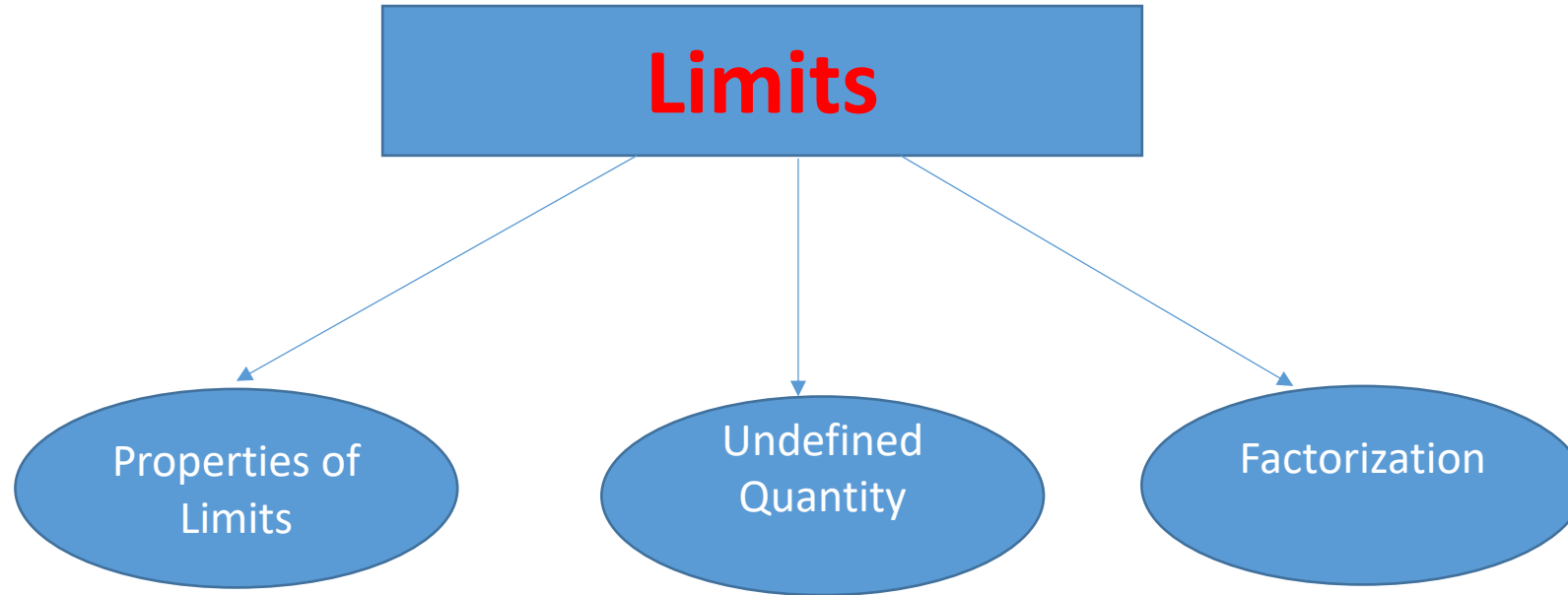
Solution: Since $\lim_{x \rightarrow 7} f(x) = \lim_{x \rightarrow 7} 5 = 5$,

$$\lim_{x \rightarrow 7} f(x) = 5 = f(7).$$

Solution: $\lim_{x \rightarrow -4} g(x) = \lim_{x \rightarrow -4} (x^2 - 3)$

$$= g(-4)$$

$$= (-4)^2 - 3 = 13$$



Continuity

1. $f(x)$ exists
 2. $\lim_{x \rightarrow a} f(x)$ exists
 3. $\lim_{x \rightarrow a} f(x) = f(a)$
-