

Mathematics 3 – MAT132

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Lecture 7 – Advanced Topics in Calculus



Lecture Objective

- ☐ To develop and apply the formula for integration by parts.
- ☐ To show how to integrate a proper rational function.
- ☐ To illustrate the use of the table of integrals.
- ☐ To develop the concept of the average value of a function.
- ☐ To define and evaluate the improper integral.
- ☐ To discuss functions of several variables and to compute function values.
- ☐ To compute partial derivatives.
- ☐ To find partial derivatives of a function defined implicitly.
- ☐ To compute higher-order derivatives.

Lecture Outline

- 1- Integration by Parts
- 2- Integration with Initial Conditions
- 3- Definite Integral

1- Integration by Parts

Formula for Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

Extra Practice - Example 1

Example 1 – Integration by Parts

Find $\int \frac{\ln x}{\sqrt{x}} dx$ by integration by parts.

Solution:

$$\begin{aligned} \text{Let } u = \ln x \text{ and } dv &= \frac{1}{\sqrt{x}} dx \\ du &= \frac{1}{x} dx & v &= \int x^{-1/2} dx = 2x^{1/2} \end{aligned}$$

$$\begin{aligned} \text{Thus, } \int \frac{\ln x}{\sqrt{x}} dx &= (\ln x)(2\sqrt{x}) - \int (2x^{1/2})\left(\frac{1}{x} dx\right) \\ &= 2\sqrt{x}[\ln(x) - 2] + C \end{aligned}$$

Extra Practice - Example 2

Example 1 – Integration by Parts

Find $\int \frac{\ln x}{\sqrt{x}} dx$ by integration by parts.

Solution:

$$\begin{aligned} \text{Let } u = \ln x \text{ and } dv &= \frac{1}{\sqrt{x}} dx \\ du &= \frac{1}{x} dx \quad v = \int x^{-1/2} dx = 2x^{1/2} \end{aligned}$$

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** Revision – Fundamentals

$$\frac{1}{\frac{1}{2}}$$

$$= 1 \div \frac{1}{2}$$

$$= 1 * 2 = 2$$

Extra Practice - Example 3

Example – Integration by Parts where u is the Entire Integrand

Determine $\int \ln y \, dy$.

Solution: Let $u = \ln y$ and $dv = dy$

$$du = \left(\frac{1}{y}\right) dy \quad v = y$$

$$\begin{aligned} \text{Thus, } \int \ln y \, dy &= (\ln y)(y) - \int y \left(\frac{1}{y} dy\right) \\ &= y \ln y - y + C \\ &= y[\ln y - 1] + C \end{aligned}$$

2- Integration with Initial Conditions

Integration with initial Conditions

- Use initial conditions to find the constant, C .

Extra Practice - Example 4

Example 1 - Initial-Condition Problem

If y is a function of x such that $y' = 8x - 4$ and $y(2) = 5$, find y .

Solution: We find the integral,

$$y = \int (8x - 4) dx = (8) \frac{x^2}{2} - 4x + C = 4x^2 - 4x + C$$

Using the condition, $5 = 4(2)^2 - 4(2) + C$

$$C = -3$$

The equation is $y = 4x^2 - 4x - 3$

Extra Practice - Example 5

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Extra Practice – Example 6

Find (y) subject to the given condition:

$$1) y' = 3x - 4 \qquad y(-1) = \frac{13}{2}$$

$$y = \int (3x - 4) dx = \frac{3x^2}{2} - 4x + C$$

$$\text{Using } y(-1) = \frac{13}{2} \text{ gives}$$

$$\frac{13}{2} = \frac{3(-1)^2}{2} - 4(-1) + C$$

$$\frac{13}{2} = \frac{11}{2} + C$$

$$\text{Thus } C = 1, \text{ so } y = \frac{3x^2}{2} - 4x + 1.$$

Extra Practice – Example 7

$$y' = \frac{9}{8\sqrt{x}}$$

$$y(16) = 10$$

$$\begin{aligned} y &= \int \frac{9}{8\sqrt{x}} dx \\ &= \frac{9}{8} \int x^{-1/2} dx \\ &= \frac{9}{8} \cdot \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= \frac{9\sqrt{x}}{4} + C \end{aligned}$$

$$y(16) = 10 \text{ implies } 10 = \frac{9\sqrt{16}}{4} + C, \quad 10 = 9 + C,$$

$$C = 1. \text{ Thus } y = \frac{9\sqrt{x}}{4} + 1.$$

Extra Practice – Example 8

$$y' = -x^2 + 2x \quad y(2) = 1$$

$$y = \int (-x^2 + 2x) dx = -\frac{x^3}{3} + x^2 + C$$

$$y(2) = 1 \text{ implies } 1 = -\frac{8}{3} + 4 + C, \text{ so } C = -\frac{1}{3}.$$

$$\text{Thus } y = -\frac{x^3}{3} + x^2 - \frac{1}{3}.$$

Extra Practice – Example 9

Find the integrals:

$$\int (x + 5)^7 dx$$

$$\text{Let } u = x + 5 \Rightarrow du = 1dx = dx$$

$$\int (x + 5)^7 [dx] = \int u^7 du = \frac{u^8}{8} + C = \frac{(x + 5)^8}{8} + C$$

Extra Practice – Example 10

$$\int 2x(x^2 + 3)^5 dx$$

$$\text{Let } u = x^2 + 3 \Rightarrow du = 2x dx$$

$$\begin{aligned}\int 2x(x^2 + 3)^5 dx &= \int (x^2 + 3)^5 [2x dx] = \int u^5 du = \frac{u^6}{6} + C \\ &= \frac{(x^2 + 3)^6}{6} + C\end{aligned}$$

Extra Practice – Example 11

$$\int \frac{4}{x} dx$$

$$\int \frac{4}{x} dx = 4 \int \frac{1}{x} dx = 4 \ln|x| + C$$

3- The Definite Integral

- For area under the graph from limit $a \rightarrow b$,

$$\int_a^b f(x) dx$$

- x is called the **variable of integration** and $f(x)$ is the **integrand**.

Extra Practice – Example 12

$$\int_3^4 \frac{3}{(x+3)^2} dx$$

Extra Practice – Example 12 Solution

$$\begin{aligned}\int_3^4 \frac{3}{(x+3)^2} dx &= 3 \int_3^4 (x+3)^{-2} dx \\ &= 3 \cdot \frac{(x+3)^{-1}}{-1} \Big|_3^4 \\ &= -3 \left[\frac{1}{4+3} - \frac{1}{3+3} \right] \\ &= -3 \left(\frac{1}{7} - \frac{1}{6} \right) \\ &= \frac{1}{14}\end{aligned}$$

Extra Practice – Example 13

$$\int_0^1 e^5 dx$$

$$\int_0^1 e^5 dx = e^5 x \Big|_0^1 = e^5 - 0 = e^5$$

Extra Practice – Example 14

$$\int_{\frac{1}{2}}^{\frac{3}{2}} (x^2 + x + 1) dx$$

Extra Practice – Example 15

$$\int_2^3 \frac{3}{x^2} dx$$

Extra Practice – Example 15 solution

$$\begin{aligned}\int_2^3 \frac{3}{x^2} dx &= 3 \int_2^3 x^{-2} dx \\&= 3 \cdot \frac{x^{-1}}{-1} \Big|_2^3 \\&= -\frac{3}{x} \Big|_2^3 \\&= \frac{-3}{3} - \left(\frac{-3}{2} \right) \\&= -1 + \frac{3}{2} \\&= \frac{1}{2}\end{aligned}$$

Extra Practice – Example 16

$$\int_{-2}^{-1} (3w^2 - w - 1) dw$$

Extra Practice – Example 16 solution

$$\int_{-2}^{-1} (3w^2 - w - 1) dw$$

$$= \left(w^3 - \frac{w^2}{2} - w \right) \Big|_{-2}^{-1}$$

$$= -\frac{1}{2} - (-8) = \frac{15}{2}$$

Extra Practice – Example 17

$$\int_2^8 -5x \, dx$$

$$\int_2^8 -5x \, dx$$

$$= -5 \cdot \frac{x^2}{2} \Big|_2^8 = -160 - (-10) = -150$$

Extra Practice – Example 18

$$\int_{-3}^1 (2x - 3) dx$$

$$\int_{-3}^1 (2x - 3) dx$$

$$= (x^2 - 3x) \Big|_{-3}^1$$

$$= -2 - 18 = -20$$

Extra Practice – Example 19

$$\int_4^1 (2t - 3t^2) dt$$

$$\int_4^1 (2t - 3t^2) dt$$

$$= (t^2 - t^3) \Big|_4^1$$

$$= 0 - (-48) = 48$$

Extra Practice – Example 20

$$\int_0^3 5dx$$

$$\int_0^3 5 dx = 5x \Big|_0^3$$

$$= 5(3) - 5(0)$$

$$= 15 - 0 = 15$$

Extra Practice – Example 21

$$\int 5x^{-7} dx$$

$$\begin{aligned}\int 5x^{-7} dx &= 5 \int x^{-7} dx = 5 \cdot \frac{x^{-7+1}}{-7+1} + C \\ &= 5 \cdot \frac{x^{-6}}{-6} + C = -\frac{5}{6x^6} + C\end{aligned}$$

Extra Practice – Example 22

$$\int 5x^{24} dx$$

$$\begin{aligned}\int 5x^{24} dx &= 5 \int x^{24} dx = 5 \cdot \frac{x^{24+1}}{24+1} + C \\ &= 5 \cdot \frac{x^{25}}{25} + C = \frac{x^{25}}{5} + C\end{aligned}$$

Extra Practice – Example 23

$$\int x^8 dx$$

$$\int x^8 dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$$

Extra Practice – Example 24

$$\int \frac{5}{x^7} dx$$

$$\begin{aligned}\int \frac{5}{x^7} dx &= 5 \int x^{-7} dx = 5 \cdot \frac{x^{-7+1}}{-7+1} + C \\ &= \frac{5x^{-6}}{-6} + C = -\frac{5}{6x^6} + C\end{aligned}$$

Extra Practice – Example 25

$$\int \frac{1}{t^{7/4}} dt$$

$$\begin{aligned}\int \frac{1}{t^{7/4}} dt &= \int t^{-7/4} dt = \frac{t^{-7/4+1}}{-\frac{7}{4}+1} + C = \frac{t^{-3/4}}{-\frac{3}{4}} + C \\ &= -\frac{4}{3t^{3/4}} + C\end{aligned}$$

Extra Practice – Example 26

$$\int (e^x + 3x^2 + 2x) dx$$

$$\begin{aligned}\int (e^x + 3x^2 + 2x) dx &= \int e^x dx + 3 \int x^2 dx + 2 \int x dx \\ &= e^x + 3 \cdot \frac{x^{2+1}}{2+1} + 2 \cdot \frac{x^{1+1}}{1+1} + C \\ &= e^x + x^3 + x^2 + C\end{aligned}$$

Extra Practice – Example 27

$$\int x^3(x^2 + 5x + 2)dx$$

$$\begin{aligned}\int x^3(x^2 + 5x + 2)dx &= \int (x^5 + 5x^4 + 2x^3)dx \\ &= \frac{x^6}{6} + 5 \cdot \frac{x^5}{5} + 2 \cdot \frac{x^4}{4} + C \\ &= \frac{x^6}{6} + x^5 + \frac{x^4}{2} + C\end{aligned}$$

Lecture Summary

1- Integration By Parts

2- Integration with Initial Conditions