

Mathematics

Lecture 7

MAT 112
Integration

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- To define the differential.
 - To define the anti-derivative and the indefinite integral.
 - To evaluate constants of integration.
 - To apply the formulas for $\int u^n du$, $\int e^n du$ and $\int \frac{1}{u} du$.
 - To handle more challenging integration problems.
 - To evaluate simple definite integrals.
 - To apply Fundamental Theorem of Integral Calculus.
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The Indefinite Integral

Integration with Initial Conditions

More Integration Formulas

Techniques of Integration

The Definite Integral

The Fundamental Theorem of Integral

- An **antiderivative** of a function f is a function F such that $F'(x) = f(x)$.

In differential notation, $dF = f(x)dx$.

- Integration states that

$$\int f(x)dx = F(x) + C \quad \text{if only} \quad F'(x) = f(x)$$

- Basic Integration

Properties:

$$1. \int k dx = kx + C \quad k \text{ is a constant}$$

$$2. \int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$$

$$3. \int x^{-1} dx = \int \frac{1}{x} dx = \int \frac{dx}{x} = \ln x + C \quad \text{for } x > 0$$

$$4. \int e^x dx = e^x + C$$

$$5. \int kf(x) dx = k \int f(x) dx \quad k \text{ is a constant}$$

$$6. \int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

Example 1 - Finding an Indefinite Integral

Find $\int 5dx$.

Solution: $\int 5dx = 5x + C$

Example - Indefinite Integral of a Constant Times a Function

Find $\int 7x dx$

Solution: $\int 7x dx = \frac{7x^2}{2} + C$

Example 5 - Finding Indefinite Integrals

a. $\int \frac{1}{\sqrt{t}} dx$

$$= \int t^{-1/2} dx = \frac{t^{1/2}}{1/2} + C = 2\sqrt{t} + C$$

b. $\int \frac{1}{6x^3} dx$

$$= \frac{1}{6} \left(\frac{x^{-3+1}}{-3+1} \right) + C = -\frac{1}{12x^2} + C$$

Example - Using Algebraic Manipulation to Find an Indefinite Integral

Find

a. $\int \frac{(2x-1)(x+3)}{6} dx$

Solution:

$$\begin{aligned} \text{a. } \int \frac{(2x-1)(x+3)}{6} dx \\ &= \frac{1}{6} \left((2) \frac{x^3}{3} + (5) \frac{x^2}{2} - 3x \right) + C \\ &= \frac{x^3}{9} + \frac{5x^2}{12} - \frac{x}{2} + C \end{aligned}$$

Example - Using Algebraic Manipulation to Find an Indefinite Integral

Find

$$\text{b. } \int \frac{x^3 - 1}{x^2} dx$$

Solution:

$$\begin{aligned} &= \int (x - x^{-2}) dx \\ &= \frac{x^2}{2} + \frac{1}{x} + C \end{aligned}$$

- Use initial conditions to find the constant, C .

Example 1 - Initial-Condition Problem

If y is a function of x such that $y' = 8x - 4$ and $y(2) = 5$, find y .

Solution: We find the integral,

$$y = \int (8x - 4) dx = (8) \frac{x^2}{2} - 4x + C = 4x^2 - 4x + C$$

Using the condition, $5 = 4(2)^2 - 4(2) + C$

$$C = -3$$

The equation is $y = 4x^2 - 4x - 3$

Power Rule for Integration

$$\int u^n dx = \frac{u^{n+1}}{n+1} + C \text{ if } n \neq -1$$

Integrating Natural Exponential Functions

$$\int e^u du = e^u + C$$

Integrals Involving Logarithmic Functions

$$\int \frac{1}{x} dx = \ln|x| + C \text{ for } x \neq 0$$

Basic Integration Formulas

$$1. \int k \, du = ku + C \quad k \text{ a constant}$$

$$2. \int u^n \, du = \frac{u^{n+1}}{n+1} + C \quad n \neq -1$$

$$3. \int \frac{1}{u} \, du = \ln |u| + C \quad u \neq 0$$

$$4. \int e^u \, du = e^u + C$$

$$5. \int kf(x) \, dx = k \int f(x) \, dx \quad k \text{ a constant}$$

$$6. \int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

Example 1 - Applying the Power Rule for Integration

Find the integral of

$$\text{a. } \int (x+1)^{20} dx$$

Solution:

$$\text{a. } \int (x+1)^{20} dx$$

$$= \frac{(x+1)^{21}}{21} + C$$

Example 1 - Applying the Power Rule for Integration

Find the integral of

b. $\int 3x^2(x^3 + 7)^3 dx$

Solution:

b. $\int 3x^2(x^3 + 7)^3 dx$

$$= \frac{(x^3 + 7)^4}{4} + C$$

Example - Integrals Involving Exponential Functions

$$\int (3x^2 + 3) e^{x^3+3x} dx$$

Solution:

$$\int (3x^2 + 3) e^{x^3+3x} dx$$

$$= e^{x^3+3x} + C$$

Example

Find $\int \sqrt[3]{6y} dy$

Solution:

$$\begin{aligned} a. \int \sqrt[3]{6y} dy \\ &= \sqrt[3]{6} \frac{y^{4/3}}{4/3} + C \\ &= \frac{3\sqrt[3]{6}}{4} y^{4/3} + C \end{aligned}$$

Example - Using Algebraic Manipulation to Find an Indefinite Integral

Find

$$\text{b. } \int \frac{x^3 - 1}{x^2} dx$$

Solution:

$$\begin{aligned} &= \int (x - x^{-2}) dx \\ &= \frac{x^2}{2} + \frac{1}{x} + C \end{aligned}$$

- For area under the graph from limit $a \rightarrow b$,

$$\int_a^b f(x)dx$$

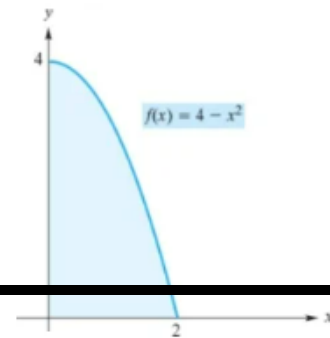
- x is called the **variable of integration** and $f(x)$ is the **integrand**.
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Example 1 - Computing an Area by Using Right-Hand Endpoints

Find the area of the region in the first quadrant bounded by $f(x) = 4 - x^2$ and the lines $x = 0$ and $y = 0$.

Solution:

Since the length of $[0, 2]$ is 2, $\Delta x = 2/n$.



Example 1 - Applying the Fundamental Theorem

Find $\int_{-1}^3 (3x^2 - x + 6)dx$.

Solution:

$$\begin{aligned}\int_{-1}^3 (3x^2 - x + 6)dx &= \left[x^3 - \frac{x^2}{2} + 6x \right]_{-1}^3 \\ &= \left[3^3 - \frac{3^2}{2} + 6(3) \right] - \left[(-1)^3 - \frac{(-1)^2}{2} + 6(-1) \right] \\ &= 48\end{aligned}$$

Example - Evaluating Definite Integrals

$$\int_0^1 e^t dt$$

Solution:

$$\int_0^1 e^t dt = \left[e^t \right]_0^1 = (e^1 - e^0) = (e^1 - 1)$$

$$\int 5x^{-7} dx$$

$$\int 5x^{-7} dx = 5 \int x^{-7} dx = 5 \cdot \frac{x^{-7+1}}{-7+1} + C$$

$$= 5 \cdot \frac{x^{-6}}{-6} + C = -\frac{5}{6x^6} + C$$

$$\int 5x^{24} dx$$

$$\begin{aligned}\int 5x^{24} dx &= 5 \int x^{24} dx = 5 \cdot \frac{x^{24+1}}{24+1} + C \\ &= 5 \cdot \frac{x^{25}}{25} + C = \frac{x^{25}}{5} + C\end{aligned}$$

$$\int x^8 dx$$

$$\int x^8 dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$$

$$\int \frac{5}{x^7} dx$$

$$\begin{aligned}\int \frac{5}{x^7} dx &= 5 \int x^{-7} dx = 5 \cdot \frac{x^{-7+1}}{-7+1} + C \\ &= \frac{5x^{-6}}{-6} + C = -\frac{5}{6x^6} + C\end{aligned}$$

$$\int \frac{1}{t^{7/4}} dt$$

$$\begin{aligned}\int \frac{1}{t^{7/4}} dt &= \int t^{-7/4} dt = \frac{t^{-7/4+1}}{-\frac{7}{4}+1} + C = \frac{t^{-3/4}}{-\frac{3}{4}} + C \\ &= -\frac{4}{3t^{3/4}} + C\end{aligned}$$

$$\int (e^x + 3x^2 + 2x) dx$$

$$\begin{aligned}\int (e^x + 3x^2 + 2x) dx &= \int e^x dx + 3 \int x^2 dx + 2 \int x dx \\ &= e^x + 3 \cdot \frac{x^{2+1}}{2+1} + 2 \cdot \frac{x^{1+1}}{1+1} + C \\ &= e^x + x^3 + x^2 + C\end{aligned}$$

$$\int x^3(x^2 + 5x + 2)dx$$

$$\begin{aligned}\int x^3(x^2 + 5x + 2)dx &= \int (x^5 + 5x^4 + 2x^3)dx \\ &= \frac{x^6}{6} + 5 \cdot \frac{x^5}{5} + 2 \cdot \frac{x^4}{4} + C \\ &= \frac{x^6}{6} + x^5 + \frac{x^4}{2} + C\end{aligned}$$

** EXTRA PRACTISE

Find (y) subject to the given condition:

$$1) y' = 3x - 4 \qquad y(-1) = \frac{13}{2}$$

$$y = \int (3x - 4) dx = \frac{3x^2}{2} - 4x + C$$

$$\text{Using } y(-1) = \frac{13}{2} \text{ gives}$$

$$\frac{13}{2} = \frac{3(-1)^2}{2} - 4(-1) + C$$

$$\frac{13}{2} = \frac{11}{2} + C$$

$$\text{Thus } C = 1, \text{ so } y = \frac{3x^2}{2} - 4x + 1.$$

$$y' = \frac{9}{8\sqrt{x}}$$

$$y(16) = 10$$

$$\begin{aligned} y &= \int \frac{9}{8\sqrt{x}} dx \\ &= \frac{9}{8} \int x^{-1/2} dx \\ &= \frac{9}{8} \cdot \frac{x^{1/2}}{\frac{1}{2}} + C \\ &= \frac{9\sqrt{x}}{4} + C \end{aligned}$$

$$y(16) = 10 \text{ implies } 10 = \frac{9\sqrt{16}}{4} + C, \quad 10 = 9 + C,$$

$$C = 1. \text{ Thus } y = \frac{9\sqrt{x}}{4} + 1.$$

$$y' = -x^2 + 2x \qquad y(2) = 1$$

$$y = \int (-x^2 + 2x) dx = -\frac{x^3}{3} + x^2 + C$$

$$y(2) = 1 \text{ implies } 1 = -\frac{8}{3} + 4 + C, \text{ so } C = -\frac{1}{3}.$$

$$\text{Thus } y = -\frac{x^3}{3} + x^2 - \frac{1}{3}.$$

Differentials

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Integration with Initial Conditions

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