

Operations Research

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IT 262
Operations Research

Lecture 10



Integer Programming

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Lecture 10 Topics

- ❖ Integer Programming
- ❖ mathematical model
- ❖ Graphical Solution
- ❖ Optimal Solution

Graphical Solution

Graphical Solution Can be used when there are two decision variables by Plot the constraint equations at their limits by converting each equation to equality, then identifying the feasible solution space and getting the optimal solution.

Graphical Solution

Steps in the graphical method :

- 1- Formulate the objective and constraint functions.
- 2- Draw a graph with one variable on the horizontal axis and one on the vertical axis.
- 3- Plot each of the constraints as if they were lines.
- 4- Outline the feasible solution space
- 5- Circle the potential solution points .These are the intersections of the constraints or axes on the inner (minimization) or outer (maximization) perimeter of the feasible solution space.
- 6- Substitute each of the potential solution point values of the two decision variables into the objective function and solve for Z .
- 7- Select the solution point that optimizes Z (Optimal Solution).

Problem 1

Dorian Auto Problem

1- The Mathematical solution to get the objective function and constraints equation

$$\text{Min } Z = 50 x_1 + 100 x_2$$

Subject to

$$7 x_1 + 2 x_2 \geq 28$$

$$2 x_1 + 12 x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

Problem 1

Solution:

1- Plot the Constraints (shown the following figure) Change constraints to equalities:

$$7X_1 + 2X_2 \geq 28$$

Line 1

$$7X_1 + 2X_2 = 28$$



$$\begin{aligned} \text{Let } X_1 &= 0 \\ 7(0) + 2X_2 &= 28 \\ X_2 &= 14 \end{aligned}$$

$X_1 \quad X_2$

The Point (0,14)

$$\begin{aligned} \text{Let } X_2 &= 0 \\ 7X_1 + 2(0) &= 28 \\ X_1 &= 4 \end{aligned}$$

$X_1 \quad X_2$

The Point (4,0)

$$2X_1 + 12X_2 \geq 24$$

Line 2

$$2X_1 + 12X_2 = 24$$



$$\begin{aligned} \text{Let } X_1 &= 0 \\ 2(0) + 12X_2 &= 24 \\ X_2 &= 2 \end{aligned}$$

$X_1 \quad X_2$

The Point (0,2)

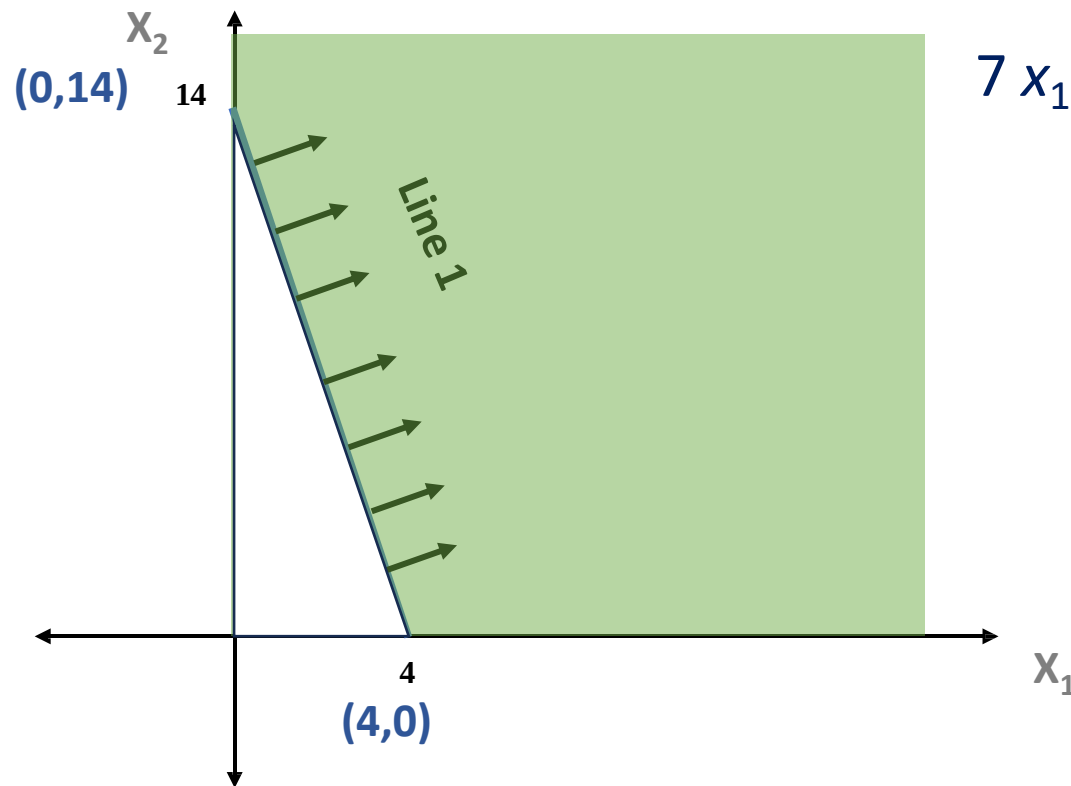
$$\begin{aligned} \text{Let } X_2 &= 0 \\ 2X_1 + 12(0) &= 24 \\ X_1 &= 12 \end{aligned}$$

$X_1 \quad X_2$

The Point (12,0)

Problem 1

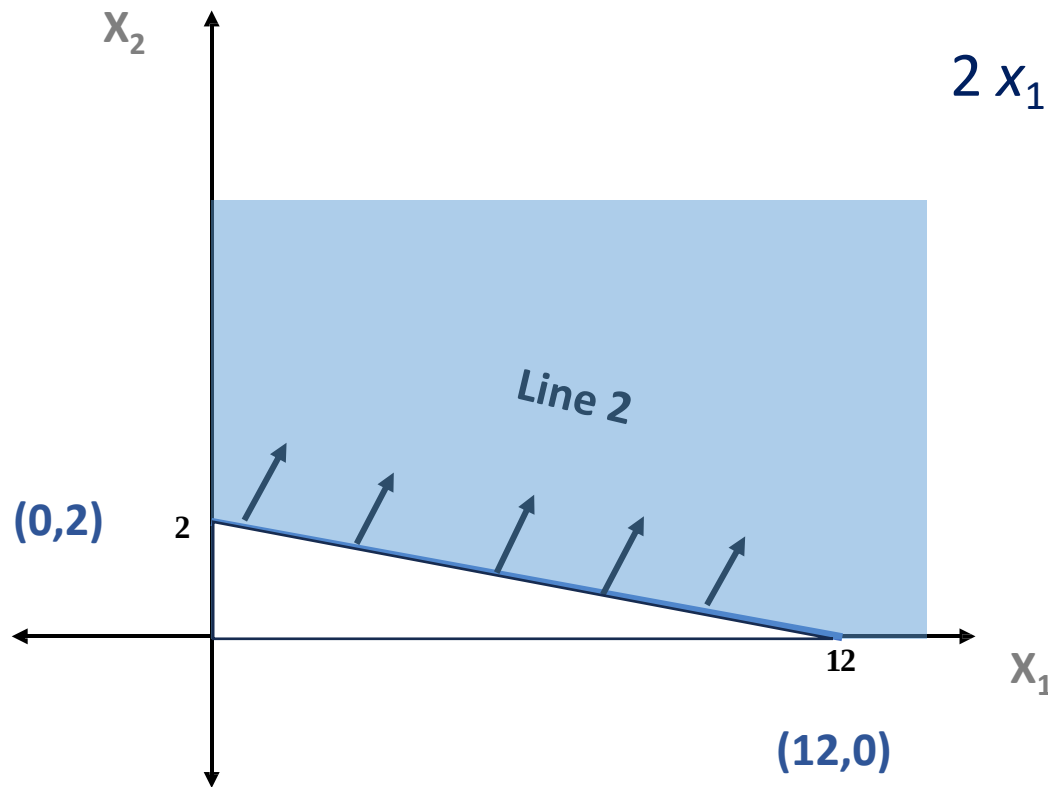
2- Graphical solution to get the Optimal Solution



$$7x_1 + 2x_2 \geq 28 \quad \text{constraint 1}$$

Problem 1

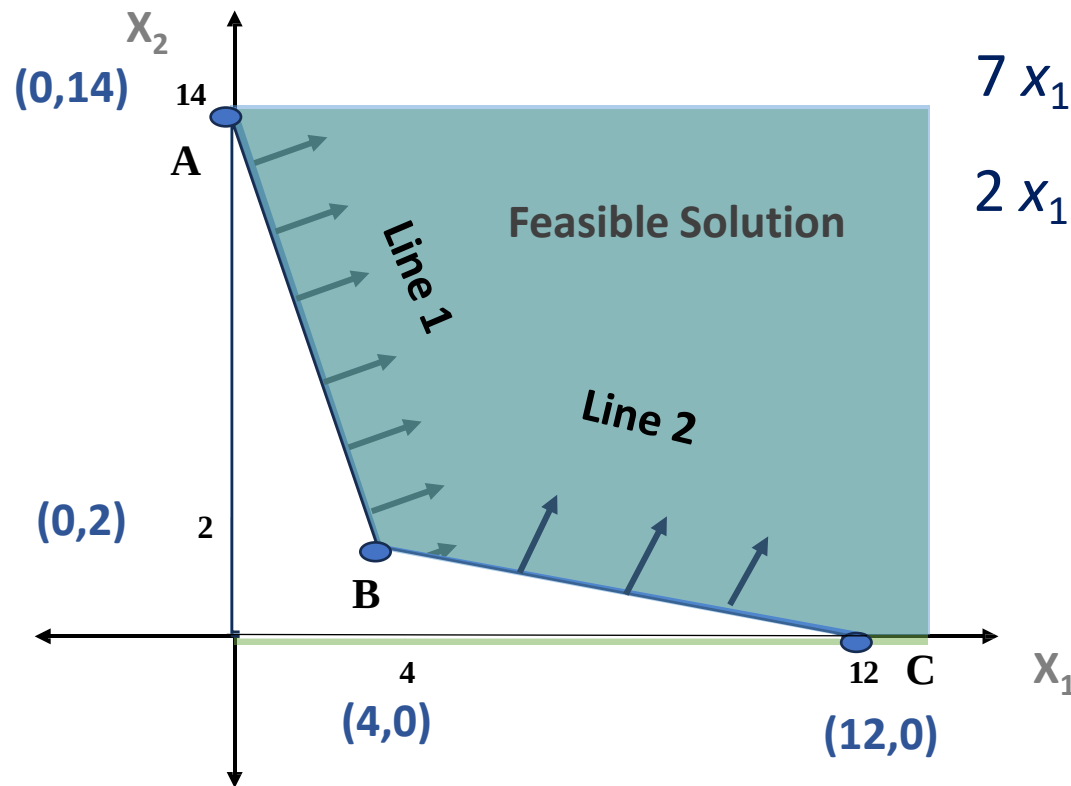
2- Graphical solution to get the Optimal Solution



$$2x_1 + 12x_2 \geq 24 \quad \text{constraint 2}$$

Problem 1

2- Graphical solution to get the Optimal Solution



$$7x_1 + 2x_2 \geq 28 \quad \text{constraint 1}$$

$$2x_1 + 12x_2 \geq 24 \quad \text{constraint 2}$$

Problem 1

To get the coordinates of point B

Line 1

$$7X_1 + 2X_2 \geq 28$$

Line 2

$$2X_1 + 12X_2 \geq 24$$



$$7X_1 + 2X_2 = 28 \quad (1)$$



$$7X_1 + 2X_2 = 28$$

$$\frac{2}{6}X_1 + 2X_2 = 4$$

$$\frac{40}{6}X_1 = 24$$

$$X_1 = 3.6$$

From equation 1:

$$7(3.6) + 2X_2 = 28$$

$$X_2 = 1.4$$

Problem 1

$$\text{Minimize } Z = 50X_1 + 100X_2$$

Point A: $(x_1=0, x_2=14)$

Point B: $(x_1=3.6, x_2=1.4)$

Point C: $(x_1=12, x_2=0)$

Points	x_1	x_2	Z
A	0	14	$Z=50(0) + 100(14) = 1400$
B	3.6	1.4	$Z=50(3.6) + 100(1.4) = 320$
C	12	0	$Z=50(12) + 100(0) = 600$

To minimize Z, the optimal solution is at point B, where $x_1=3.6$ and $x_2=1.4$ Then the $Z = 320$ \$

The optimal solution is $Z = 320$ \$

Integer Programming

Why study integer programming

- The Linear Programming divisibility assumption قابلیه وجود کسور is not always valid.
- Many different real-life situation can be modeled as integer program.
- There are efficient algorithms to solve integer programs.

Branch and Bound

- The concept of this method is that of **divide and conquer**, where we break down the large difficult problem into smaller sub-problem.
- Two mechanism:
 - A mechanism to generate a branches when searching the solution space.
 - A mechanism to generate a bound so that many branches can be terminated.

Branch and Bound

➤ These three basic steps:

- Branching
- Bounding
- Fathoming

-Branching: Choose one of the variables whose value will be fixed to create new subproblems.

Branch and Bound

- **Bounding:** A bound is optimistic estimate of the solution of a problem. Solving the linear programming (LP) relaxation of a subproblem will yield an optimistic estimate of solution.
- In case of *Minimization problem*, the value of the bound will be less than the optimal solution. ($Z_{LP} \leq Z^*$)
- In case of *Maximization problem*, the value of the bound will be greater than the optimal solution. ($Z_{LP} \geq Z^*$)

Branch and Bound

- **Fathoming criteria: (Stop)**

There are three situation that will arise that lets us know when we no longer need to break down problems.

- **A subproblem is fathomed if**
 - Its LP relaxation's optimal solution is integral
 - Its bound is worse أسوأ than the best solution found so far حل أفضل توصل إليه
 - Its LP relaxation has no feasible solution

Integer Programming

Problem 2

$$\text{Maximize } Z = 8X_1 + 5X_2$$

- The two constraints can be expressed as:

$$X_1 + X_2 \leq 6 \quad \text{constraint 1}$$

$$9X_1 + 5X_2 \leq 45 \quad \text{constraint 2}$$

$$X_1, X_2 \geq 0$$

$$X_1, X_2 \text{ integer}$$

Problem 2

Solution:

1- Plot the Constraints (shown the following figure) Change constraints to equalities:

$$X_1 + X_2 \leq 6$$

Line 1

$$X_1 + X_2 = 6$$



$$\begin{aligned} \text{Let } X_1 &= 0 \\ (0) + X_2 &= 6 \\ X_2 &= 6 \end{aligned}$$

$x_1 \quad x_2$

The Point (0 , 6)

$$\begin{aligned} \text{Let } X_2 &= 0 \\ X_1 + (0) &= 6 \\ X_1 &= 6 \end{aligned}$$

$x_1 \quad x_2$

The Point (6 , 0)

$$9X_1 + 5X_2 \leq 45$$

Line 2

$$9X_1 + 5X_2 = 45$$



$$\begin{aligned} \text{Let } X_1 &= 0 \\ 9(0) + 5 X_2 &= 45 \\ X_2 &= 9 \end{aligned}$$

$x_1 \quad x_2$

The Point (0 , 9)

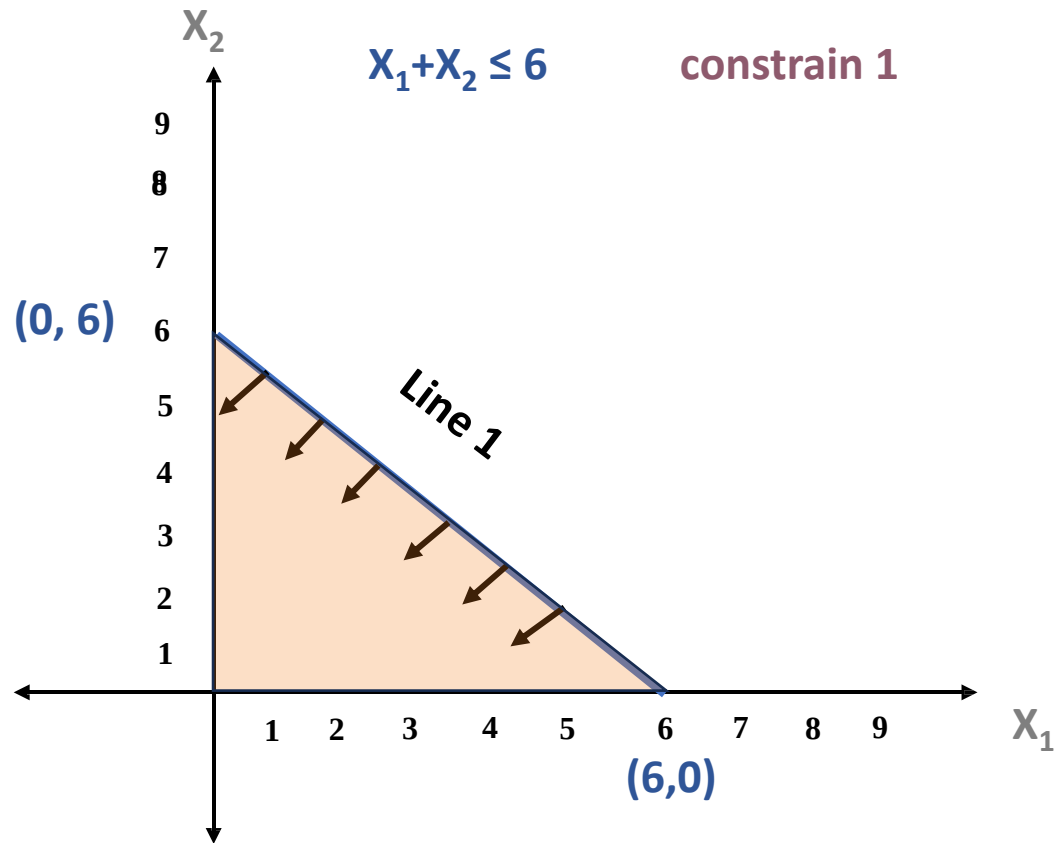
$$\begin{aligned} \text{Let } X_2 &= 0 \\ 9 X_1 + 5 (0) &= 45 \\ X_1 &= 5 \end{aligned}$$

$x_1 \quad x_2$

The Point (5 , 0)

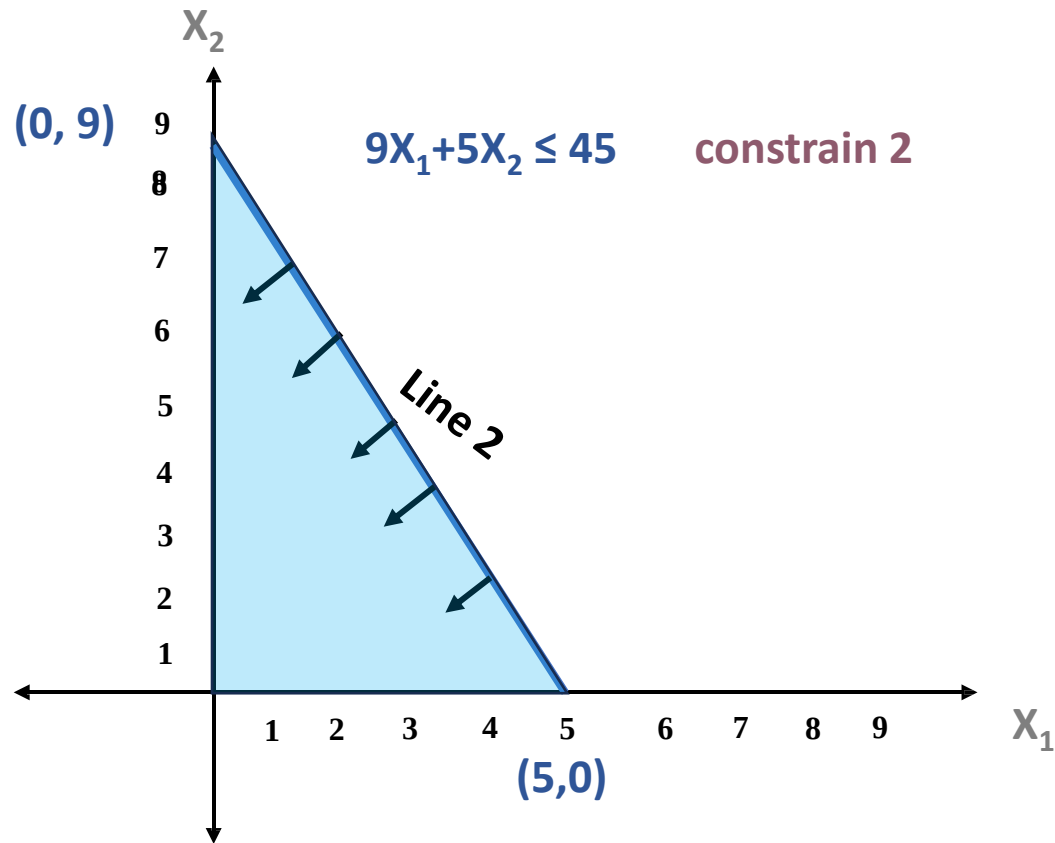
Problem 2

2- Graphical solution to get the Optimal Solution



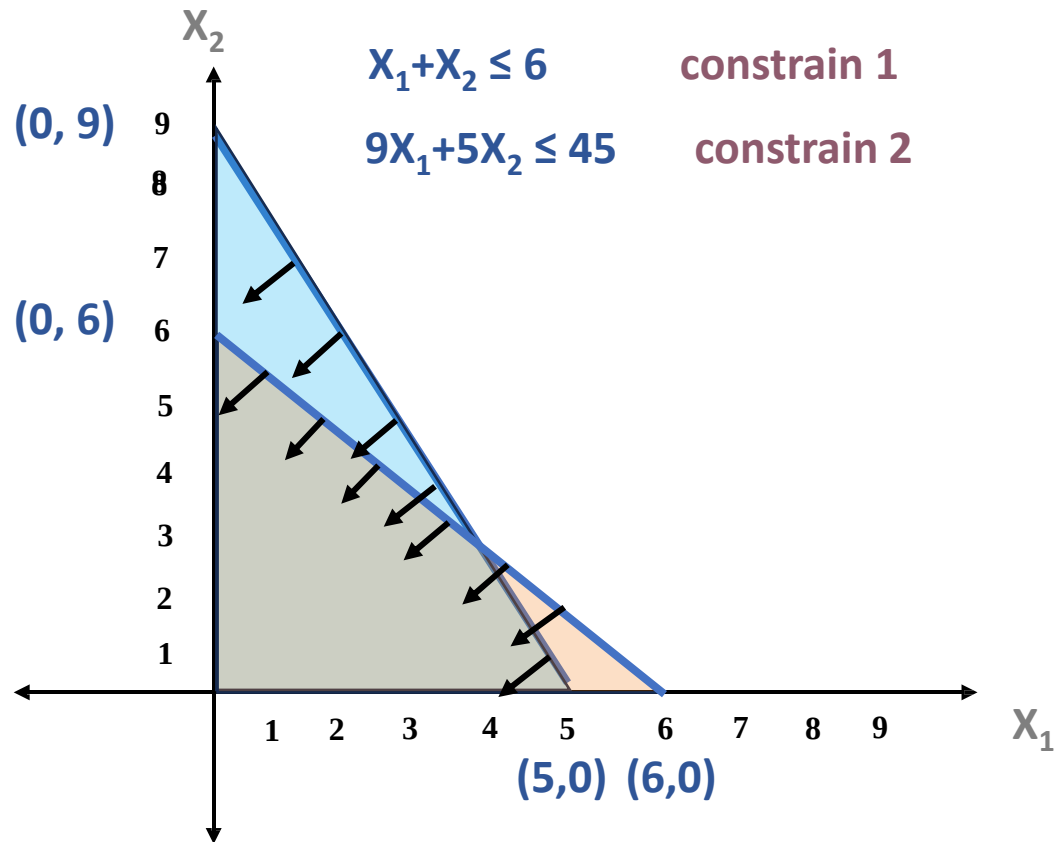
Problem 2

2- Graphical solution to get the Optimal Solution



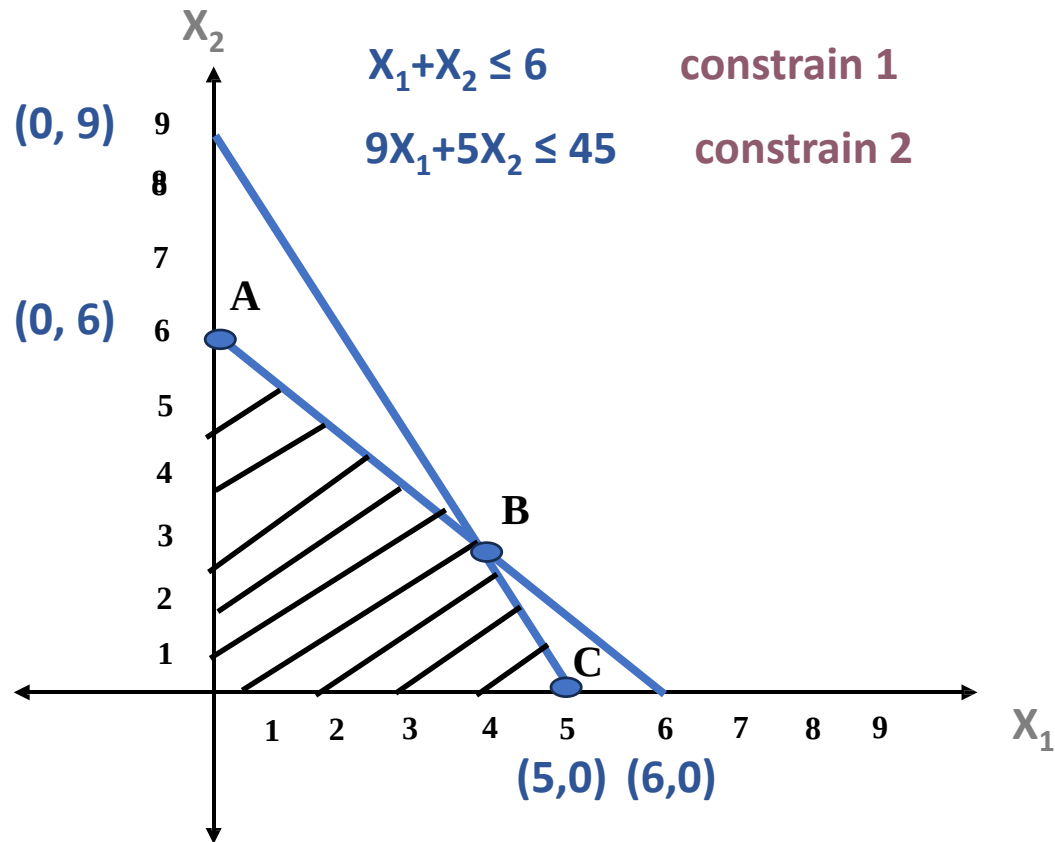
Problem 2

2- Graphical solution to get the Optimal Solution



Problem 2

2- Graphical solution to get the Optimal Solution



Problem 2

To get the coordinates of point B

Line 1

$$X_1 + X_2 \leq 6$$

Line 2

$$9X_1 + 5X_2 \leq 45$$



$$X_1 + X_2 = 6$$

$$9X_1 + 5X_2 = 45$$

$$(1) \times 9$$



$$9X_1 + 9X_2 = 54$$

$$9X_1 + 5X_2 = 45$$

$$4X_2 = 9$$

$$X_2 = 2.25$$

From equation 1:

$$X_1 + 2.25 = 6$$

$$X_1 = 3.75$$

Problem 2

$$\text{Maximize } Z = 8X_1 + 5X_2$$

Point A: $(x_1=0, x_2=6)$

Point B: $(x_1=3.75, x_2=2.25)$

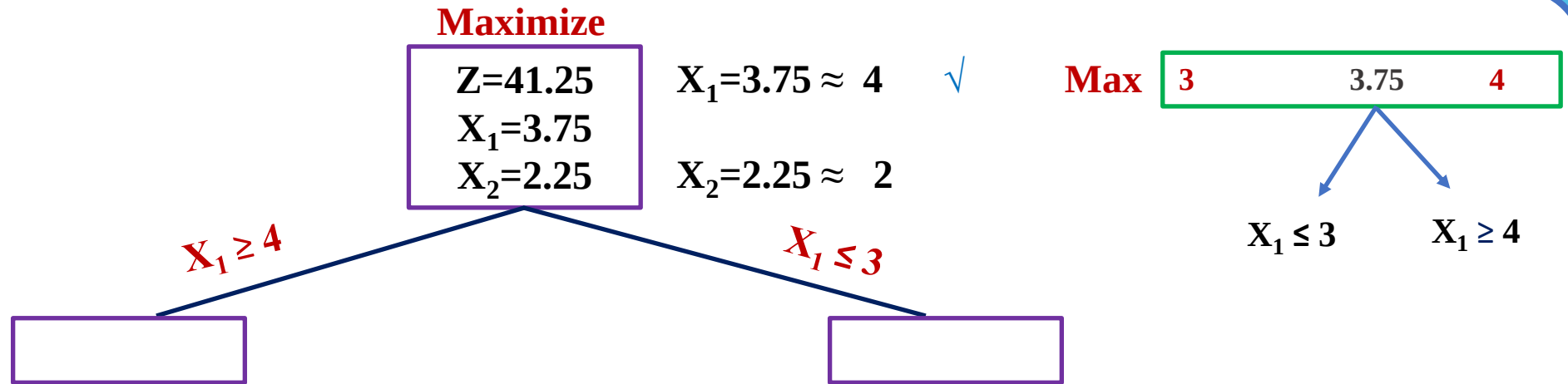
Point C: $(x_1=5, x_2=0)$

Points	x_1	x_2	Z
A	0	6	$Z=8(0) + 5(6) = 30$
B	3.75	2.25	$Z=8(3.75) + 5(2.25) = 41.25$
C	5	0	$Z=8(5) + 5(0) = 40$

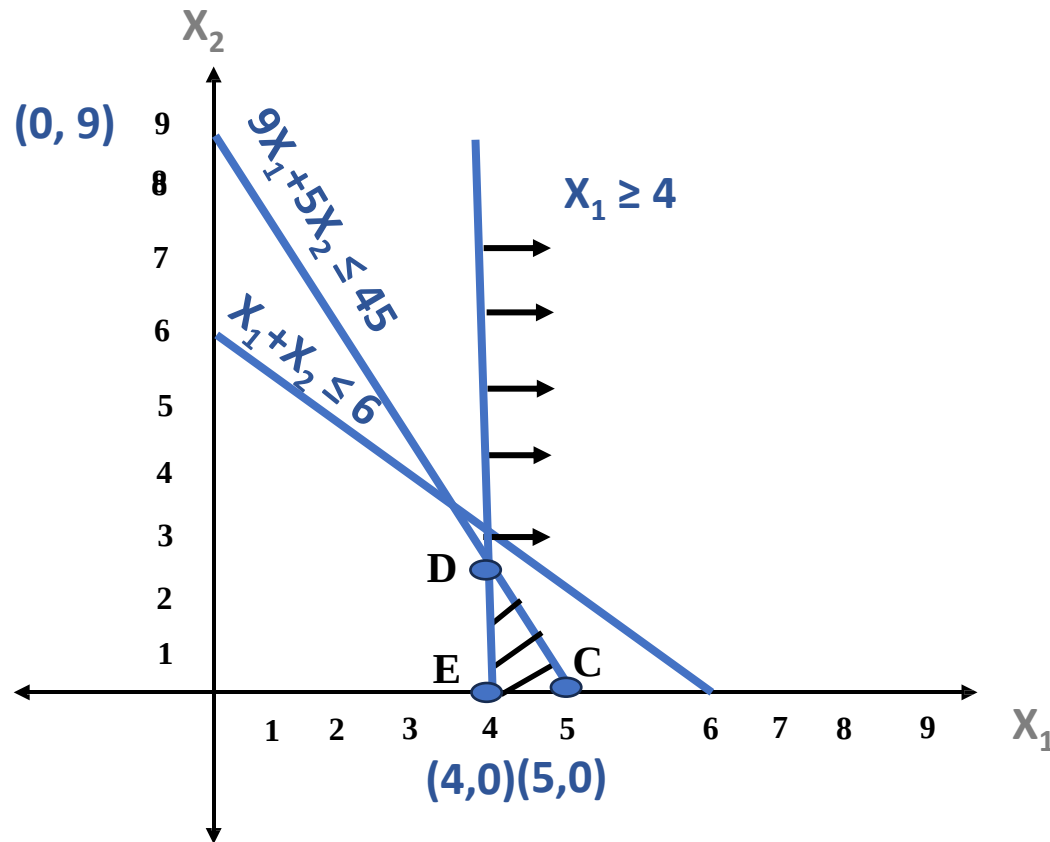
To minimize Z, the optimal solution is at point B, where $x_1=3.75$ and $x_2=2.25$ Then the $Z = 41.25$

The optimal solution is $Z = 41.25$ but X_1 and X_2 must be integer

Problem 2



Problem 2



$$X_1 + X_2 \leq 6$$

constrain 1

$$9X_1 + 5X_2 \leq 45$$

constrain 2

$$X_1 \geq 4$$

constrain 3

Problem 2

To get the coordinates of point D

Line 1

$$X_1 \geq 4$$

Line 2

$$9X_1 + 5X_2 \leq 45$$

$$X_1 = 4 \quad (1)$$

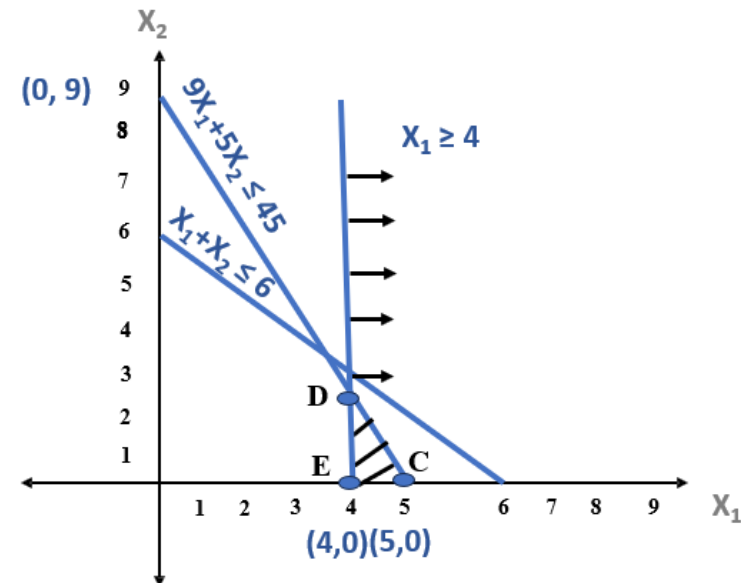
$$9X_1 + 5X_2 = 45 \quad (2)$$

$$X_1 = 4$$

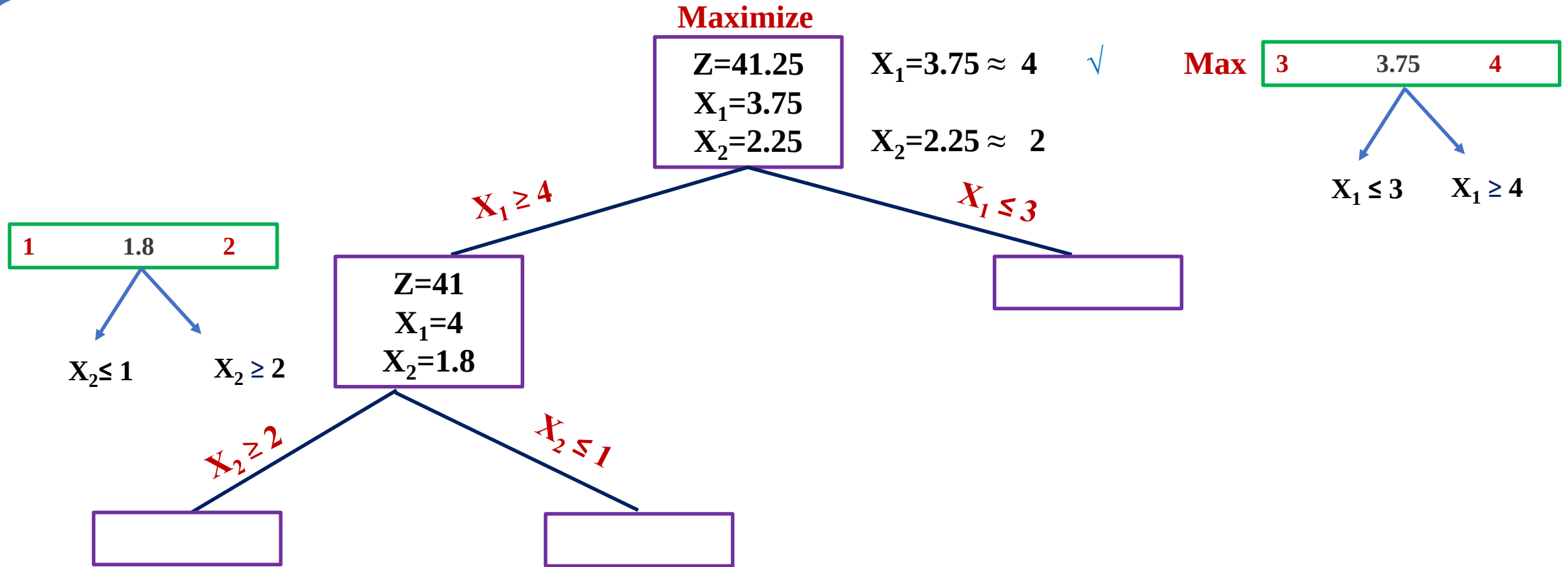
From equation 2:

$$X_2 = 1.8$$

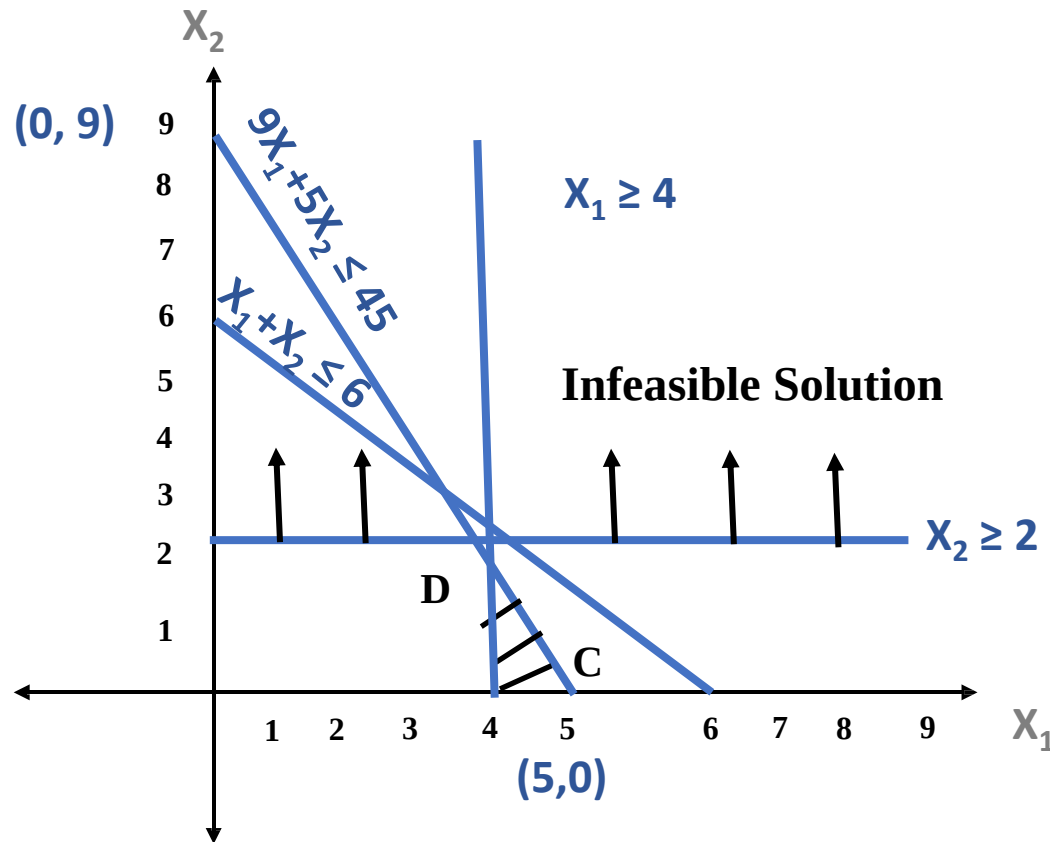
Points	x_1	x_2	Z
D	4	1.8	$Z = 8(4) + 5(1.8) = 41$
C	5	0	$Z = 8(5) + 5(0) = 40$
E	4	0	$Z = 8(4) + 5(0) = 32$



Problem 2



Problem 2



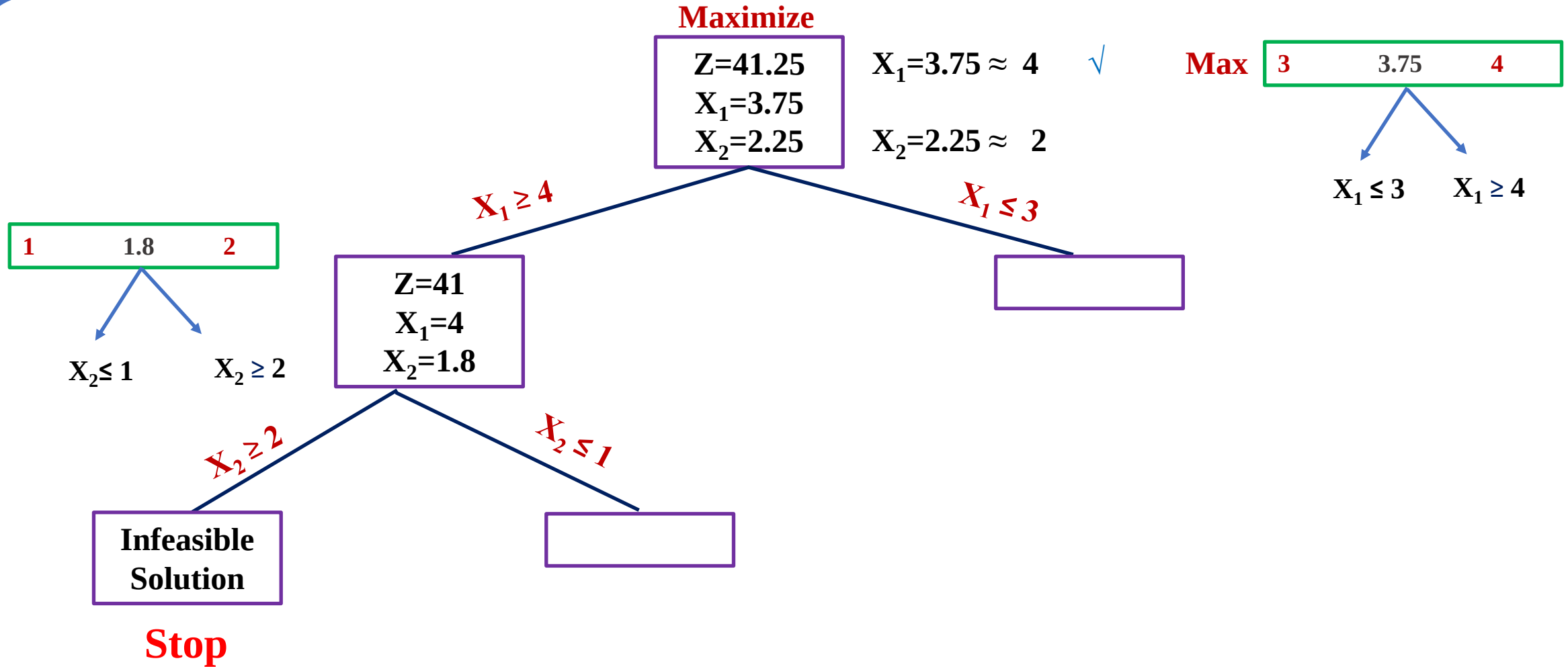
$$X_1 + X_2 \leq 6 \quad \text{constrain 1}$$

$$9X_1 + 5X_2 \leq 45 \quad \text{constrain 2}$$

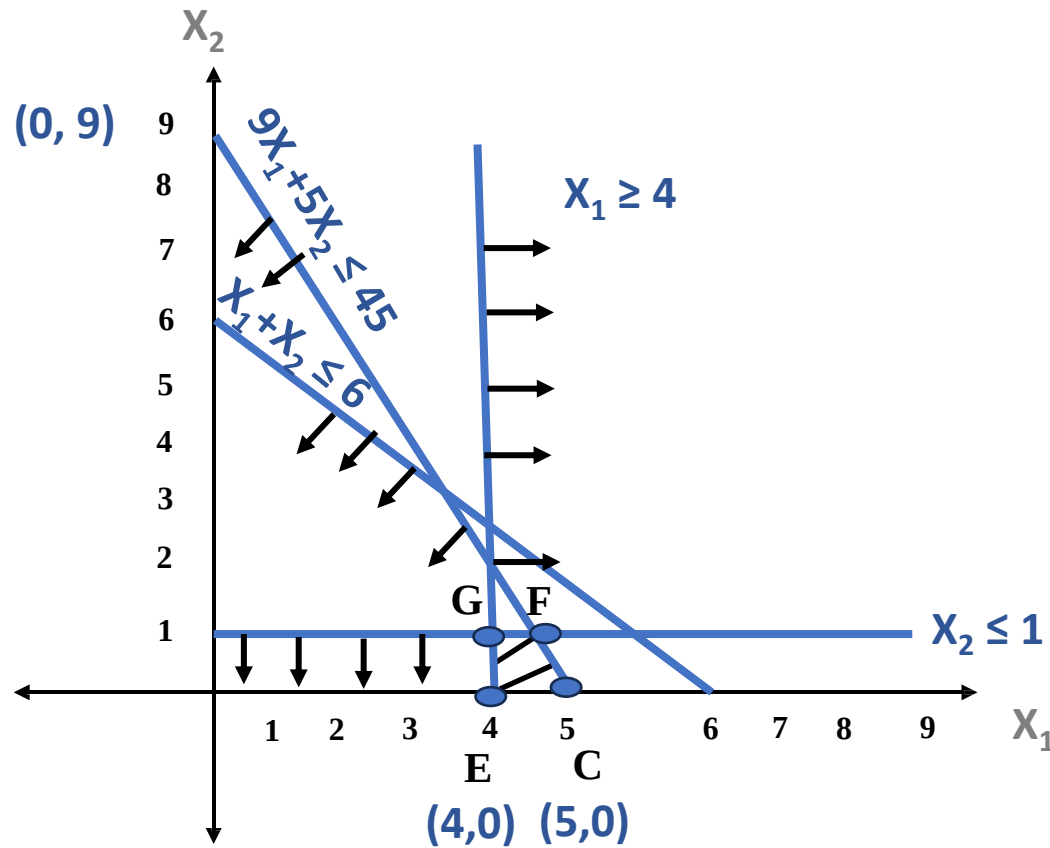
$$X_1 \geq 4 \quad \text{constrain 3}$$

$$X_2 \geq 2 \quad \text{constrain 4}$$

Problem 2



Problem 2



$$X_1 + X_2 \leq 6 \quad \text{constrain 1}$$

$$9X_1 + 5X_2 \leq 45 \quad \text{constrain 2}$$

$$X_1 \geq 4 \quad \text{constrain 3}$$

$$X_2 \leq 1 \quad \text{constrain 4}$$

Problem 2

To get the coordinates of point F

Line 4

$$X_2 \leq 1$$

Line 2

$$9X_1 + 5X_2 \leq 45$$

$$X_2 = 1 \quad (1)$$

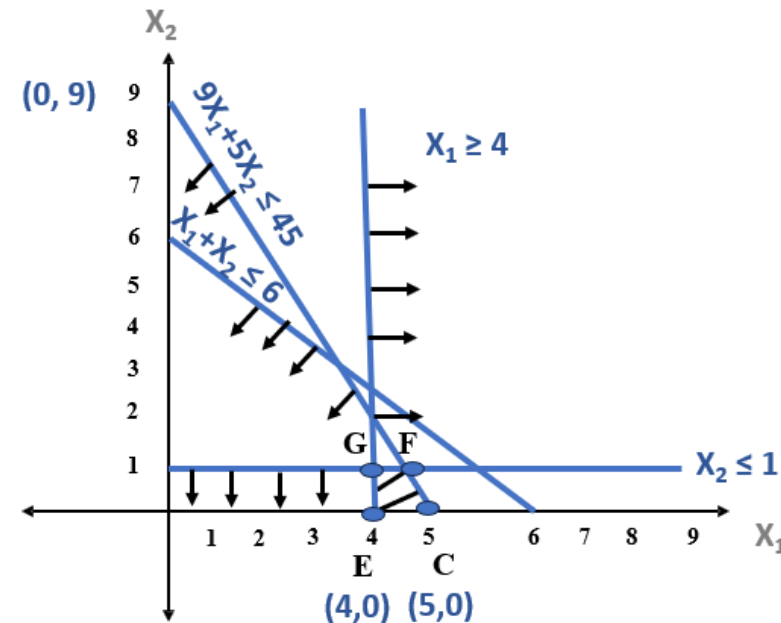
$$9X_1 + 5X_2 = 45 \quad (2)$$

$$X_2 = 1$$

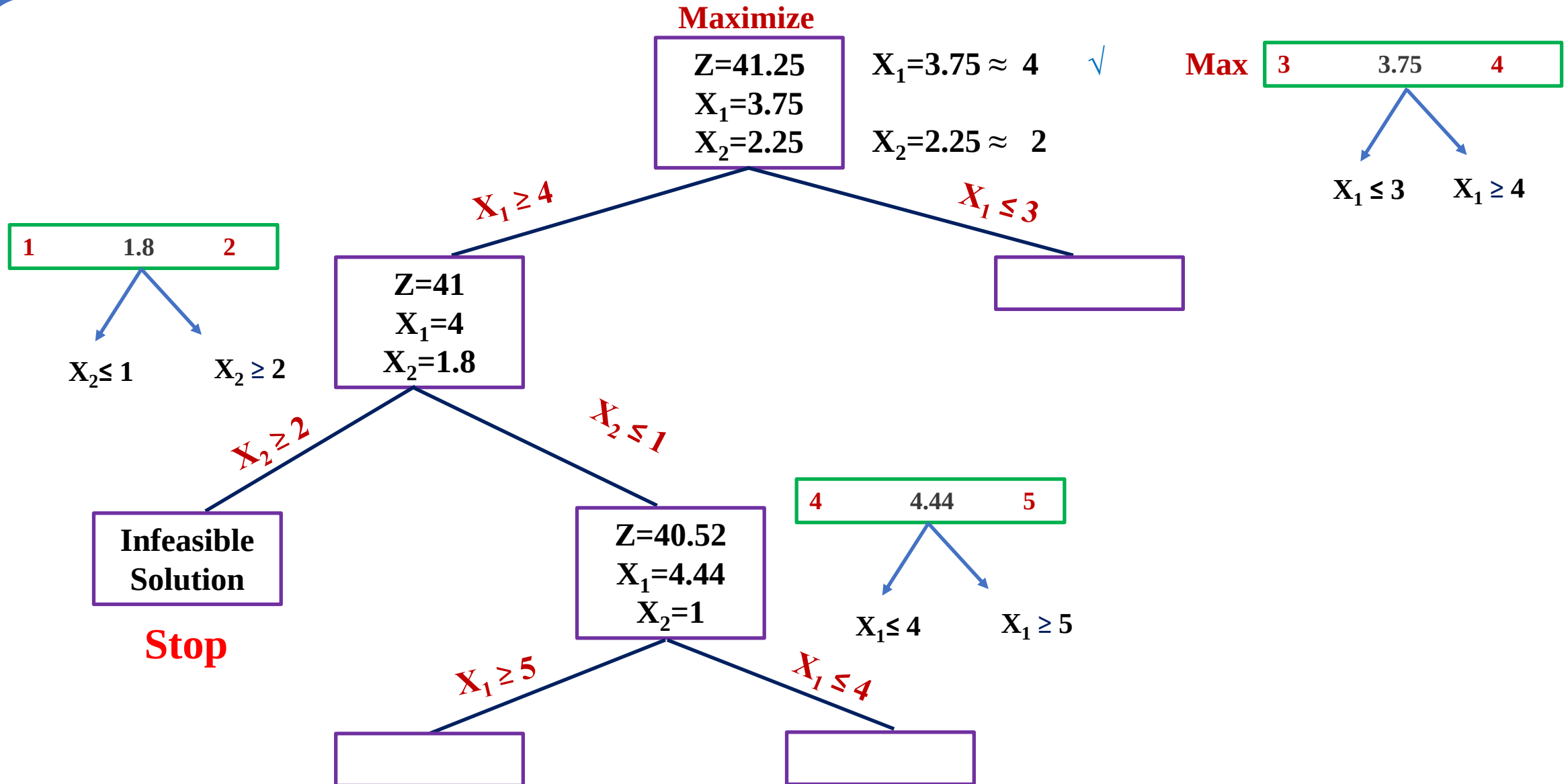
From equation 2:

$$X_1 = 4.44$$

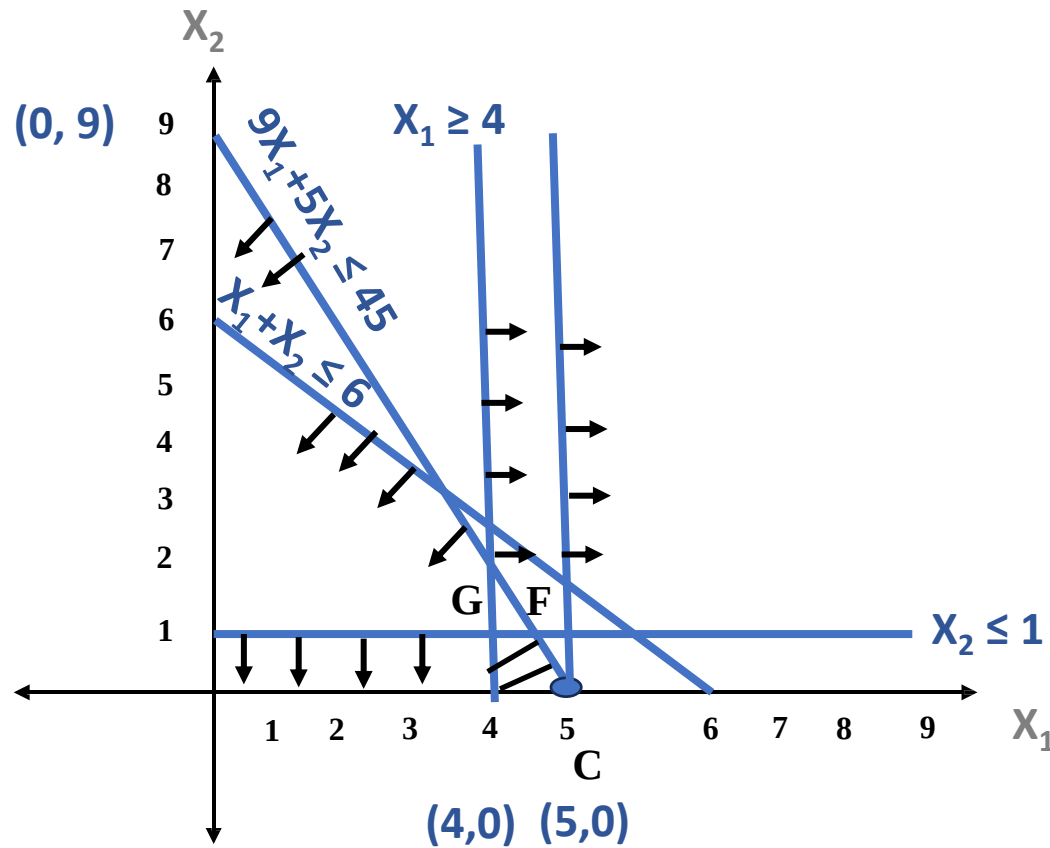
Points	x_1	x_2	Z
F	4.44	1	$Z = 8(4.44) + 5(1) = 40.52$
C	5	0	$Z = 8(5) + 5(0) = 40$
E	4	0	$Z = 8(4) + 5(0) = 32$
G	4	1	$Z = 8(4) + 5(1) = 37$



Problem 2



Problem 2



$$X_1 + X_2 \leq 6$$

constrain 1

$$9X_1 + 5X_2 \leq 45$$

constrain 2

$$X_1 \geq 4$$

constrain 3

$$X_2 \leq 1$$

constrain 4

$$X_1 \geq 5$$

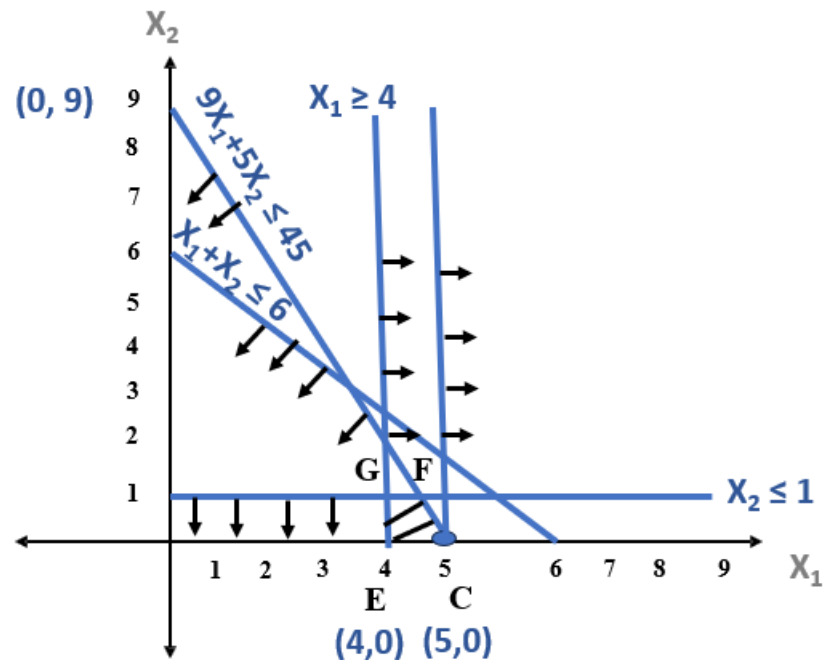
constrain 5

Problem 2

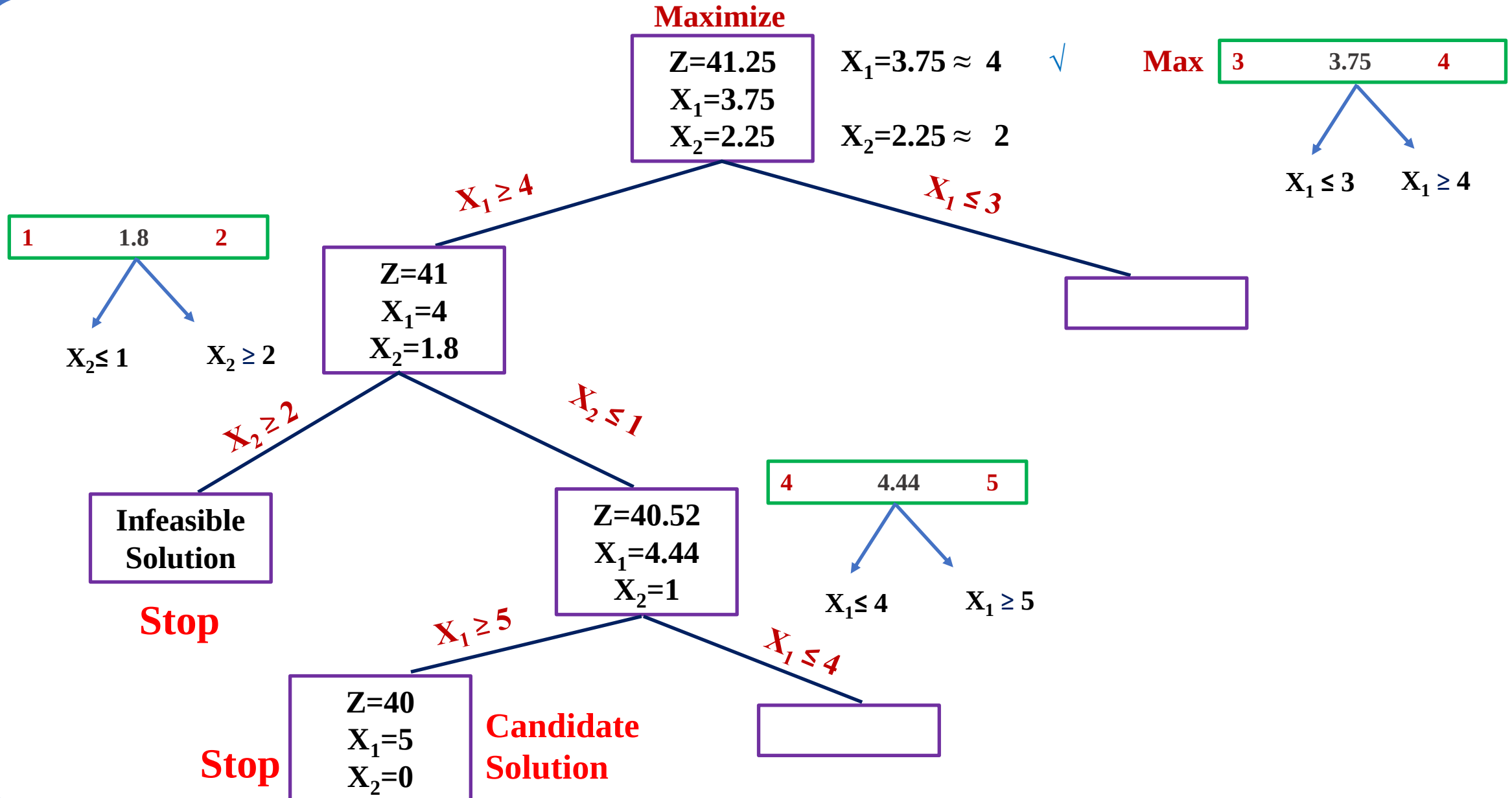
Points	x_1	x_2	Z
C	5	0	$Z=8(5) + 5(0) = 40$

$$x_1 = 5$$

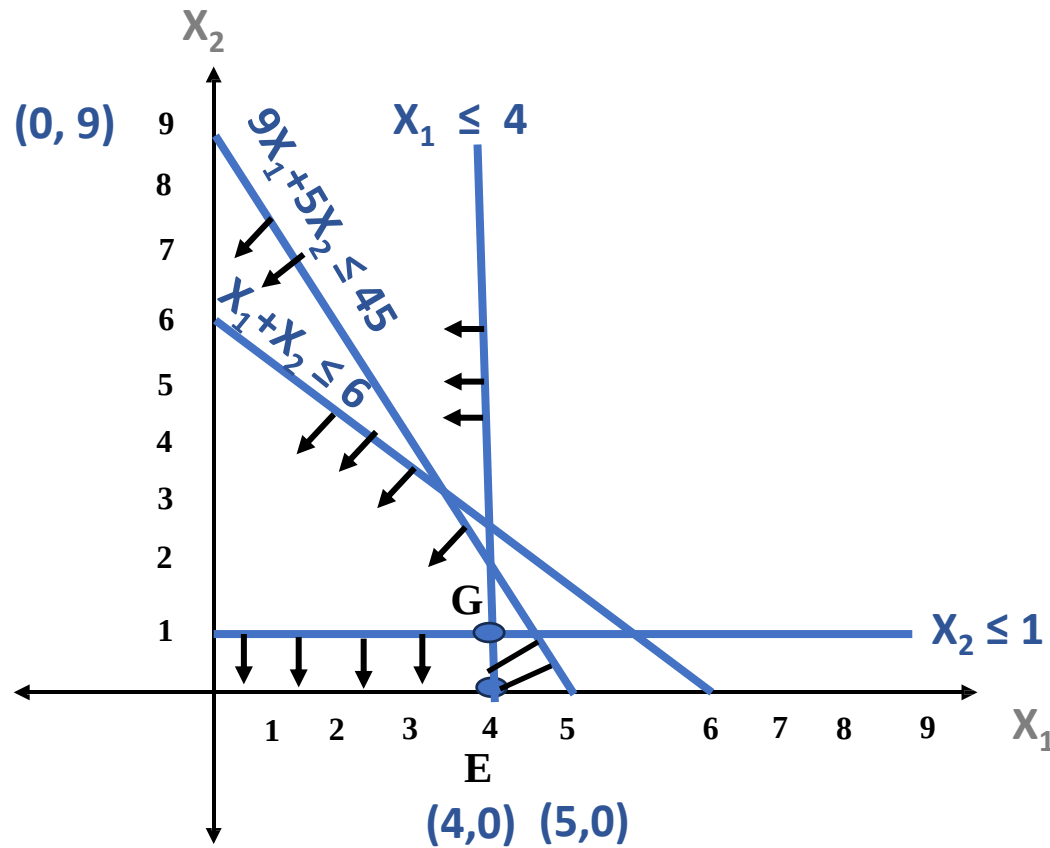
$$x_2 = 0$$



Problem 2



Problem 2



$$X_1 + X_2 \leq 6$$

constrain 1

$$9X_1 + 5X_2 \leq 45$$

constrain 2

$$X_1 \geq 4$$

constrain 3

$$X_2 \leq 1$$

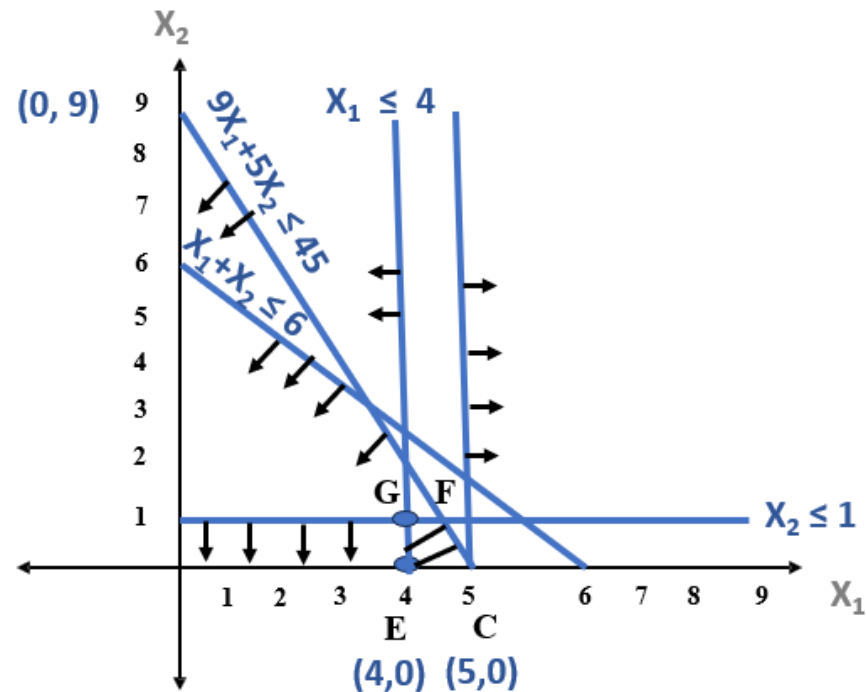
constrain 4

$$X_1 \leq 4$$

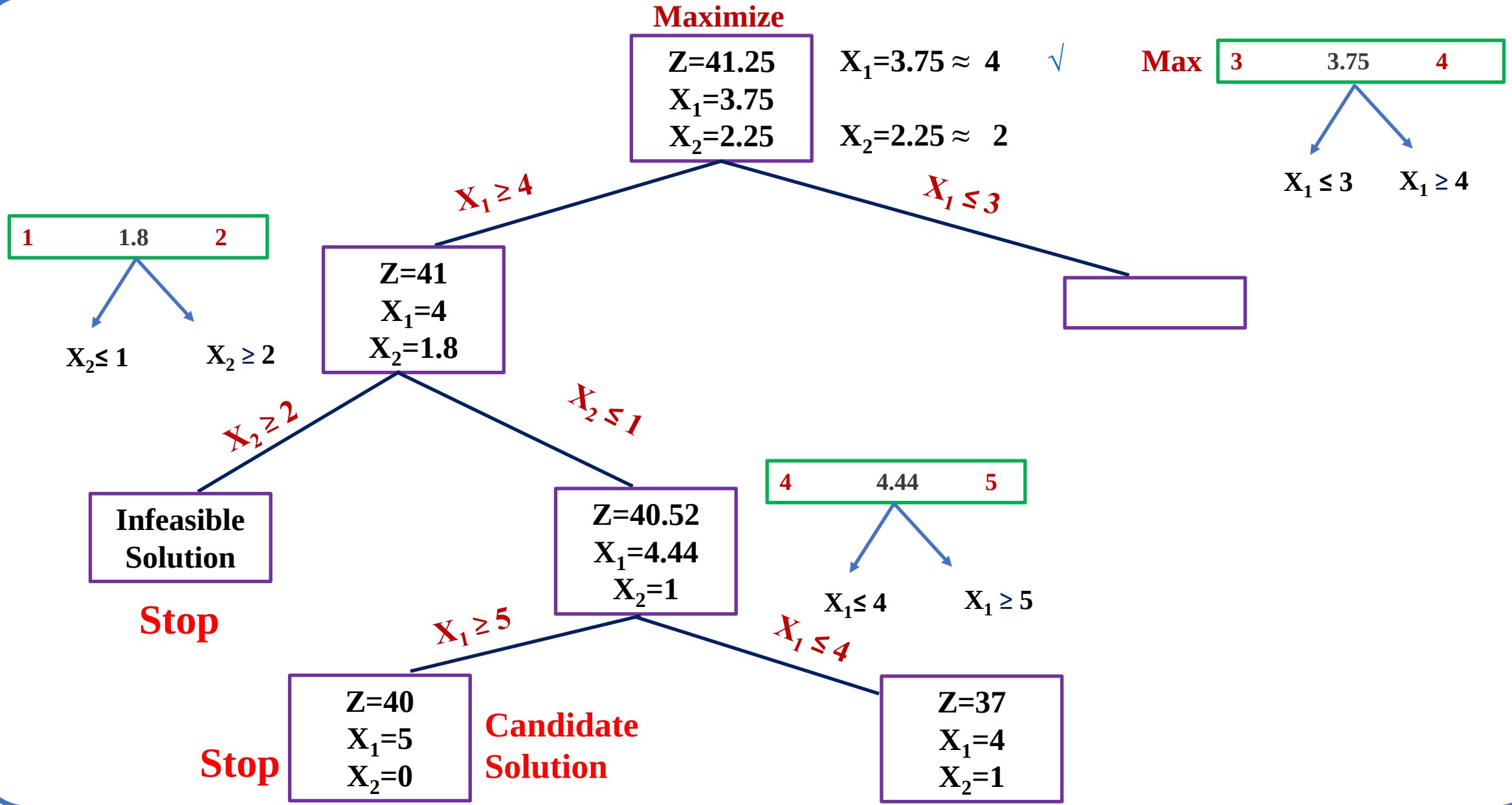
constrain 5

Problem 2

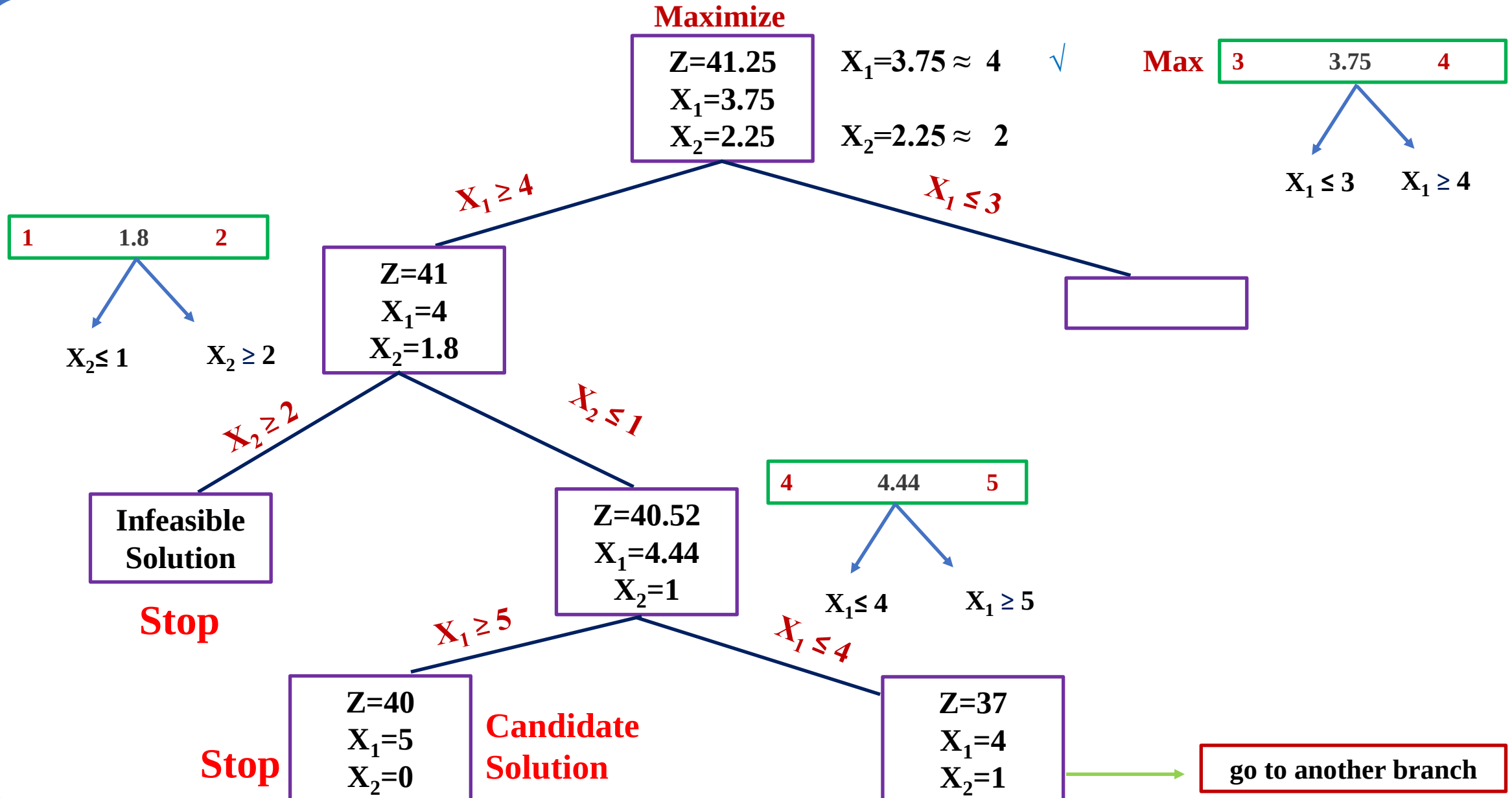
Points	x_1	x_2	Z
E	4	0	$Z=8(4) + 5(0) = 32$
G	4	1	$Z=8(4) + 5(1) = 37$



Problem 2

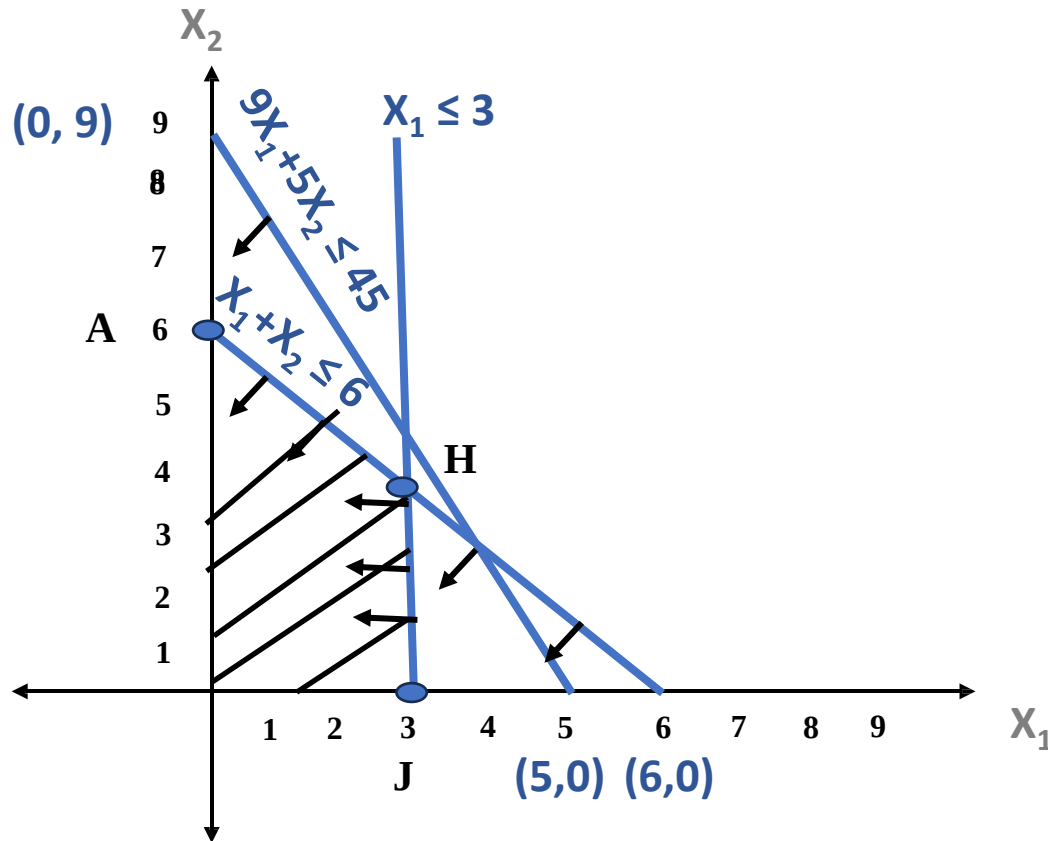


Problem 2



Problem 2

2- Graphical solution to get the Optimal Solution



$$X_1 + X_2 \leq 6$$

constrain 1

$$9X_1 + 5X_2 \leq 45$$

constrain 2

$$X_1 \leq 3$$

constrain 3

Problem 2

To get the coordinates of point H

Line 4

$$X_1 \leq 3$$

Line 2

$$X_1 + X_2 \leq 6$$

$$X_1 = 3 \quad (1)$$

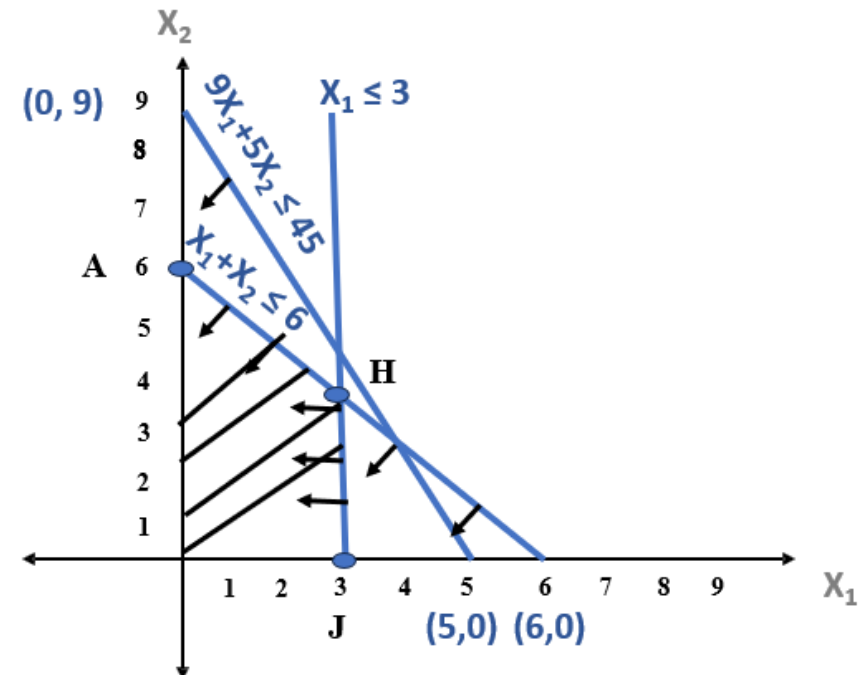
$$X_1 + X_2 = 6 \quad (2)$$

$$X_1 = 3$$

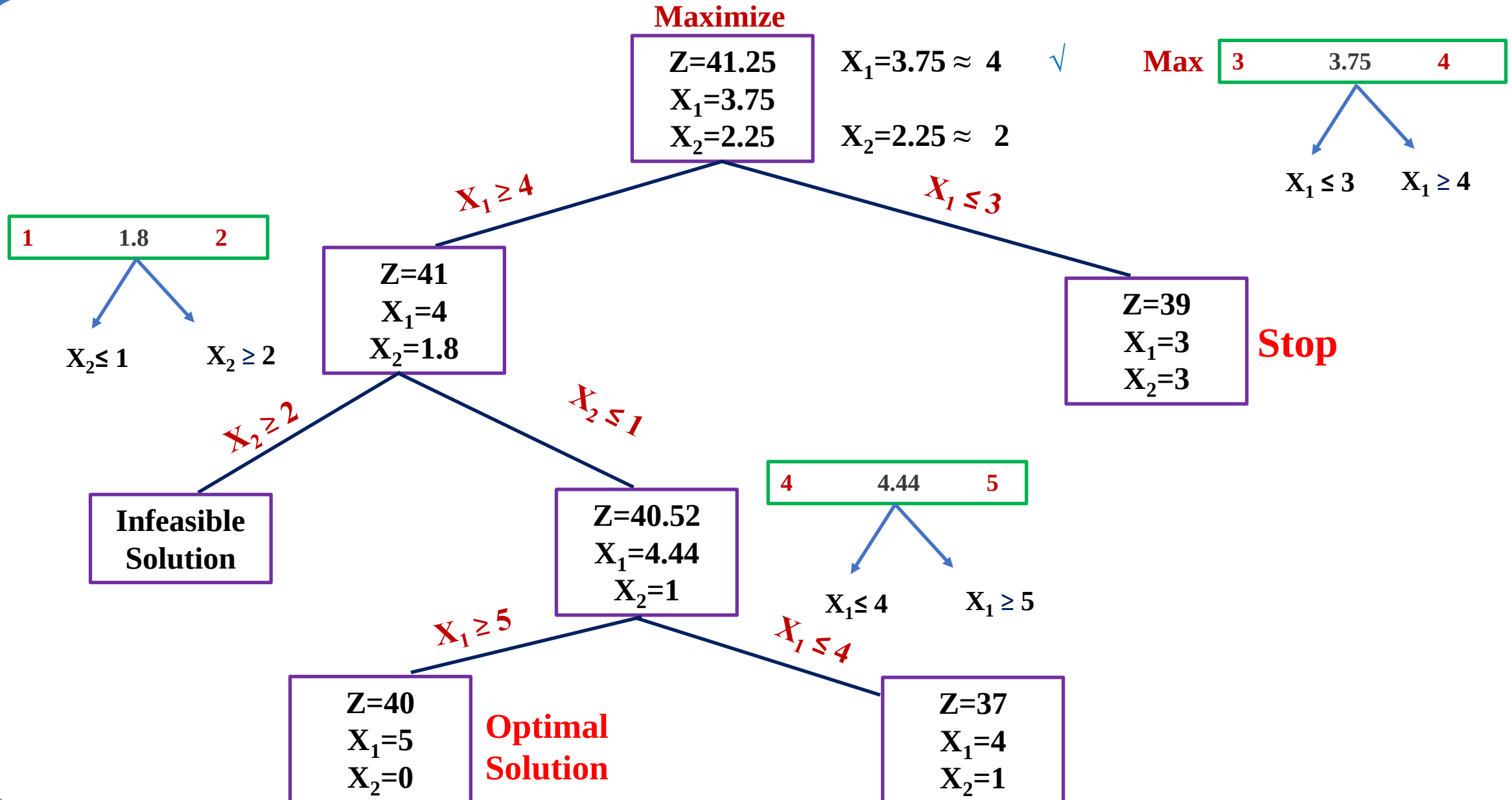
From equation 2:

$$X_2 = 3$$

Points	x_1	x_2	Z
H	3	3	$Z = 8(3) + 5(3) = 39$
A	0	6	$Z = 8(0) + 5(6) = 30$
J	3	0	$Z = 8(3) + 5(0) = 24$



Problem 2



Problem 2

$$\text{Maximize } Z = 8X_1 + 5X_2$$

**To maximize Z, in the integer programming
the optimal solution is where $x_1=5$ and $x_2=0$ Then the $Z = 40$**

The optimal solution is $Z = 40$



Thank You