

Financial mathematics

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Lecture 08



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MTH 112
Business Mathematics

Chapter 05

At the end of this section you should be familiar with:

- Compound Interest
- Continuous compounding interest
- Annual Percentage rate APR
- Depreciation

When interest is compounded several times per year

In reality, interest may be compounded several times per year, for example it may be compounded daily, weekly, monthly, quarterly, semi-annually or continuously. Each time period is known as a conversion period or interest period. The number of conversion periods per year is denoted by the symbol m ; the interest rate applied at each conversion is i/m .

daily $m = 365$
 $\rightarrow 360$
 $\rightarrow 365$

semi-annually $m = 2$
 quarterly $m = 4$
 monthly $m = 12$
 weekly $m = 52$

i interest-rate
 I interest

When interest is compounded several times per year

For example, an investment compounded twelve times per year will have twelve conversion periods; therefore if a five-year investment was compounded twelve times annually, then the investment would have sixty conversion periods; that is,

$$n = m \times t$$

where

n = total number of conversion periods

m = conversion periods per year

t = number of years

When interest is compounded several times per year

The value of the investment at the end of n conversion periods is

$$P_t = P_0 \left(1 + \frac{i}{m} \right)^n = P_0 \left(1 + \frac{i}{m} \right)^{m \times t} \quad (5.13)$$

Principle

$$n = m \times t$$

Example:

£5000 is invested for three years at 8% per annum compounded semi-annually.

- (a) Calculate the total value of the investment. $P_t = P_3$
- (b) Compare the return on the investment when interest is compounded annually to that when compounded semi-annually.
- (c) Calculate the total value of the investment when compounded, (i) monthly, (ii) daily. Assume all months consist of 365/12 days.

Solution: (a)

$$\begin{aligned}P_t &= P_0 \left(1 + \frac{i}{m}\right)^n \\&= (5000) \left(1 + \frac{0.08}{2}\right)^6 \\&= (5000) (1.04)^6 \\&= 6326.59\end{aligned}$$

$$\begin{aligned}n &= m \times t \\n &= (2)(3) \\&= 6\end{aligned}$$

Solution: (b) for the investment compounded annually

$$\begin{aligned} P_3 &= P_0 (1+i)^t \\ &= (5000) (1.08)^3 \\ &= 6298.5 \end{aligned}$$

$$\begin{array}{r} 6326.59 \\ - 6298.5 \\ \hline \text{£ } 28.09 \end{array}$$

A gain of £28.09 over the value when compounded annually

Solution: (c) Monthly compounding $m=12$

$$P_3 = P_0 \left(1 + \frac{i}{m} \right)^n$$

$$= (5000) \left(1 + \frac{0.08}{12} \right)^{36}$$

$$= (5000) (1.2702)$$

$$= 6351$$

$$n = m \times t$$

$$= (12)(3)$$

$$= 36$$

compounded Annually	6298.5
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This earns $6351 - 6298.5 = £52.5$ more than when interest is compounded annually.



Solution:

(c) Daily compounding

$$P_3 = (5000) \left(1 + \frac{0.08}{365} \right)^{1095}$$

$$= (5000) (1.2712)$$

$$= 6356$$

$$m = 365$$

$$n = m \times t$$

$$= (365)(3)$$

$$= 1095$$

This earns $6356 - 6298.5 = \text{£ } 57.5$ more than when interest is compounded annually.

Continuous compounding

In the previous example, compounding over several time intervals was illustrated: semiannually, monthly and daily. In each case, the return on the investment increases as m , the number of compounding per year, increases. One could also compound continuously, for example every minute, second or microsecond. With continuous compounding, the total value is given by the formula)

$$P_t = P_0 \left(1 + \frac{i}{m}\right)^n$$

$$P_t = P_0 e^{it} \quad (5.14)$$

where $e = 2.718281828$.

Continuous compounding

This formula is derived from formula (5.13), by letting the number of compoundings per year, m , become very large so that

$$P_t = P_0 \left(1 + \frac{i}{m} \right)^{(m)(t)} = P_0 \left[\left(1 + \frac{i}{m} \right)^m \right]^t = P_0 [e^i]^t = P_0 e^{it}$$

Since $\left(1 + \frac{i}{m} \right)^m$ approaches e^i as m gets larger.

Continuous compounding

$$P_t = P_0 \left(1 + \frac{i}{m}\right)^{(m)(t)} = P_0 \left[\left(1 + \frac{i}{m}\right)^m \right]^t = P_0 [e^i]^t = P_0 e^{it}$$

To check this by evaluating $\left(1 + \frac{i}{m}\right)^m$ for one value of i and various values of m , such as

$$i = 5/100 = 0.05,$$

$$m = 10, 20, 30, 40, 50, 365.$$

You should get, 1.05114, 1.0512055, 1.051227, 1.0512383, 1.0512448, 1.0512675

while $e^i = e^{0.05} = 1.0512711$.

Example

A financial consultant advises you to invest £5000 at 8% compounded continuously for three years.

Find the total value of your investment.

Solution

compounded continuously

$$\begin{aligned}P_3 &= P_0 e^{it} \\&= (5000) e^{(0.08)(3)} \\&= (5000) (1.27) \\&= 6356.24\end{aligned}$$

$$P_0 = 5000$$

$$i = 0.08$$

$$t = 3$$

A gain of £57.74 than compounded annually.

Annual percentage rate (APR)

Interest rates are usually cited as nominal rates of interest expressed as per annum figures. However, as compounding may occur several times during the year with the nominal rate, the amount owed or accumulated will be different from that calculated by compounding once a year.

Annual percentage rate (APR)

So, a standard measure is needed to compare the amount earned at quoted nominal rates of interest when compounding is carried out several times per year. This standard measure is called the annual percentage rate (APR) or effective annual rate.

Annual percentage rate (APR)

If interest is compounded once a year at the APR rate the investment would yield exactly the same return, P_t , at the end of t years, as it would if interest were compounded m times per year at the nominal rate.

Annual percentage rate (APR)

For example, consider an investment of £100 at a nominal interest rate of 10% compounded semi-annually, that is, at 5% per six months. The investment at the end of the first six months is worth $\text{£}[100 + 0.05(100)] = \text{£}105$. Then invest this £105 for a further six months and it grows to $\text{£}[105 + 0.05(105)] = \text{£}110.25$. At the end of the year, the initial investment of £100 has grown to £110.25, giving an annual percentage rate, APR, of 10.25% which is greater than the nominal rate of 10%.

The formula for the annual percentage rate (APR) is derived as follows:

$$P_t = P_0 \left(1 + \frac{i}{m}\right)^{mt} \quad \text{nominal rate compounded } m \text{ times per year}$$

$$P_t = P_0(1 + APR)^t \quad \text{APR rate compounded annually}$$

$$P_0(1 + APR)^t = P_0 \left(1 + \frac{i}{m}\right)^{mt} \quad \text{since the yield is the same}$$

$$(1 + APR)^t = \left(1 + \frac{i}{m}\right)^{mt} \quad \text{dividing both sides by } P_0$$

$$1 + APR = \left(1 + \frac{i}{m}\right)^m \quad \text{taking the } t\text{th root of each side}$$

$$APR = \left(1 + \frac{i}{m}\right)^m - 1$$

To explain each step in the derivation of the formula for the APR when interest is compounded continuously as outlined below:

$$P_t = P_0 e^{it} \quad \text{and} \quad P_t = P_0 (1 + APR)^t$$

$$P_0 (1 + APR)^t = P_0 e^{it}$$

$$(1 + APR)^t = e^{it}$$

$$1 + APR = e^i$$

$$APR = e^i - 1$$

The APR, therefore, may be calculated by the following formulae:

- When the nominal rate is compounded m times per year.

$$APR = \left(1 + \frac{i}{m}\right)^m - 1$$

- When the nominal rate is compounded continuously.

$$APR = e^i - 1$$

Example:

APR

$$i = 6\% = \frac{6}{100} = 0.06$$
$$m = 12$$

- a). Find the annual percentage rate on loan corresponding to 6.0% compounded monthly.
 - b). Two banks in a local town quote the following nominal interest rates: bank A, Pays interest on a savings account at 6.60% compounded monthly and bank B, pays 6.65% on a savings account compounded semi-annually.
- Which bank pays its savers the most interest?

a).

$$\text{APR} = \left(1 + \frac{i}{m}\right)^m - 1 = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 1.06167 - 1 = 6.17\%$$

The annual percentage rate payable on the loan is 6.17% which is greater than the quoted nominal rate of 6.0%.

b).

$$i = 6.6\% \quad m = 12$$

Bank **A** pays 6.60% compounded monthly. The annual percentage rate is given by

$$APR = \left(1 + \frac{i}{m}\right)^m - 1 = \left(1 + \frac{0.066}{12}\right)^{12} - 1 = 1.0680 - 1 = 0.0680 = \underline{6.8\%}$$

b).

$$i = 6.65\% \quad m = 2$$

Bank **B** pays 6.65% compounded semi-annually. The annual percentage rate is given by

$$APR = \left(1 + \frac{i}{m}\right)^m - 1 = \left(1 + \frac{0.0665}{2}\right)^2 - 1 = 1.0676 - 1 = 0.0676 = \underline{6.76\%}$$

B.

Bank **A** offers the greater annual percentage rate and thus pays its savers more interest despite its lower nominal interest rate.

At the end of this section, you should be able to:

- Calculate straight-line depreciation
- Calculate reducing-balance depreciation

Straight-line depreciation

Straight-line depreciation is the converse of simple interest with equal amounts being subtracted from the original asset value each year.

Straight-line depreciation

For example,

if the original value of a machine was £20000 and after four years its value is estimated to be £8000, then the amount of straight-line depreciation subtracted each year is $(£20000 - £8000) \div 4 = £3000$.

Reducing-balance depreciation:

Reducing-balance depreciation is the converse of compound interest with larger amounts being subtracted from the original asset value each year.

The formula for reducing balance depreciation is given as

$$A_t = A_0 (1 - i)^t \quad (5.17)$$

Where

A_t = Value of the asset after t years taking account of depreciation

A_0 = Original value of the asset

i = depreciation rate expressed as a decimal (e . g . 5% is 0 . 05)

t = number of time periods, usually years .

Example:

A machine costing £30 000 depreciates by 15% each year.
Calculate the value of the machine after five years. What is the total amount of depreciation?

A_0

$i = 15\%$

$t = 5$



Solution:

Write down any information given:

$$A_0 = \text{£}30,000$$

$$i = 15/100 = 0.15$$

$$t = 5$$

Substitute these values into formula (5.17) to calculate the book value of the machine after five years:

$$A_t = A_0 (1 - i)^t = 30\,000(1 - 0.15)^5 = \text{£}13,311.16$$

Solution:

After five years the book value of the machine is £13,311.16 Therefore.

The total amount of depreciation is $(30,000 - 13,311.16) = £16,688.84$

Example:

Five year, after purchase, a computer has a scrap value of £300.
The depreciation rate is 26%. Calculate the value of the computer when it was bought five years ago.

Solution:

In this example, you are required to calculate the original value of the computer when first bought.

Write down any information given:

$$\begin{aligned}A_t &= 300, \\ i &= 26\% = 0.26, \\ t &= 5\end{aligned}$$

Solution:

Substitute these values into formula (5.17)

$$A_t = A_0 (1 - i)^t$$

$$A_0 = \frac{A_t}{(1 - i)^t}$$

rearranging

$$A_0 = \frac{300}{(1 - 0.26)^5} = \frac{300}{0.2219} = 1351.96$$

$$A_t = A_0 (1 - i)^t$$

$$300 = A_0 (1 - 0.26)^5$$

$$\frac{300}{(1 - 0.26)^5} = A_0$$



Solution:

The original value of the computer five years ago was £1351.96

Thank you ..