

Mathematics 3

Lecture 6

MAT 132
Fourier Series

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Lecture Objective

- Essential information and Important notes for the Fourier series
- Even function and odd function
- Periodic Function
- Fourier series
- Extra Practice

Formula of $\sin x \sin y$

$$\sin x \sin y = \frac{\cos(x - y) - \cos(x + y)}{2}$$

Formula of $\cos x \cos y$

$$\cos x \cos y = \frac{\cos(x + y) + \cos(x - y)}{2}$$

Even and Odd functions

Even Functions

If a function is such that $f(-x) = f(x)$ for all values of x , it is called an **even function**.

For example, show that $f(x) = 3x^4 - x^2 + 2$ is an even function.

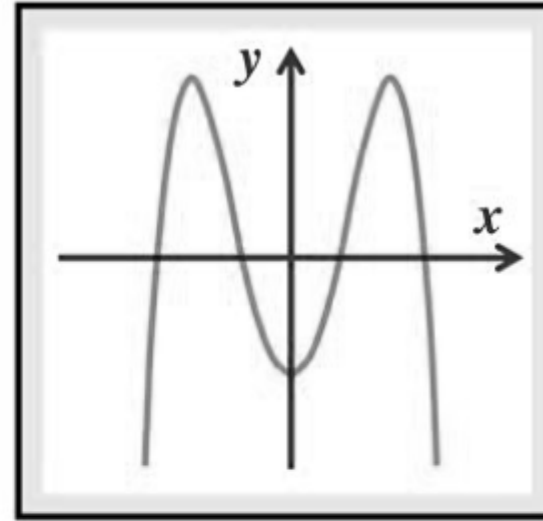
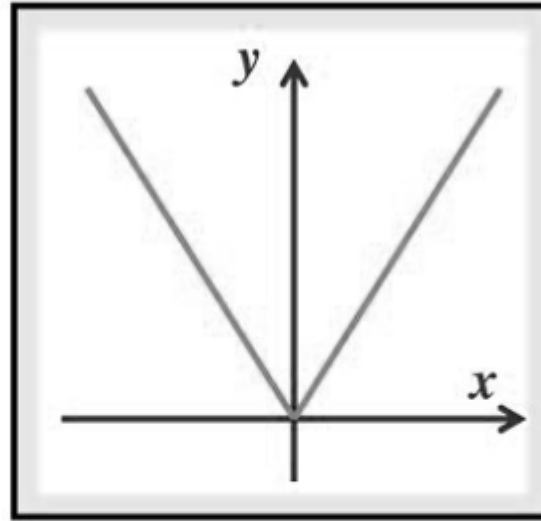
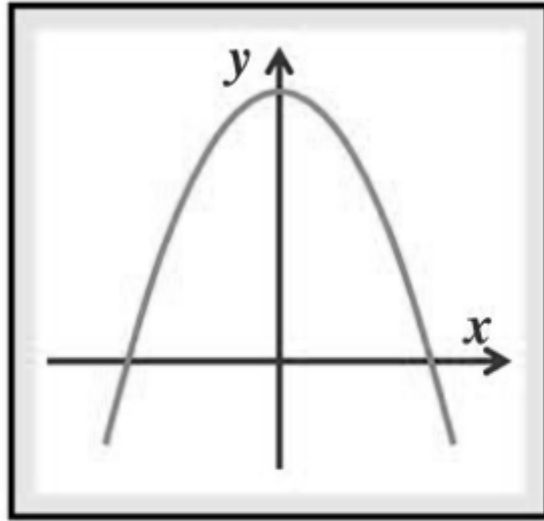
$$f(-x) = 3(-x)^4 - (-x)^2 + 2$$

$$= 3x^4 - x^2 + 2$$

$$= f(x)$$

$f(-x) = f(x)$, so the function is even.

Graphs of Even Functions



What do you notice about these graphs?

The graphs of all even functions are symmetric about the y -axis.

Odd Functions

If a function is such that $f(-x) = -f(x)$ for all values of x , it is called an **odd function**.

For example, show that $f(x) = x^3 + 4x$ is an odd function.

$$f(-x) = (-x)^3 + 4(-x)$$

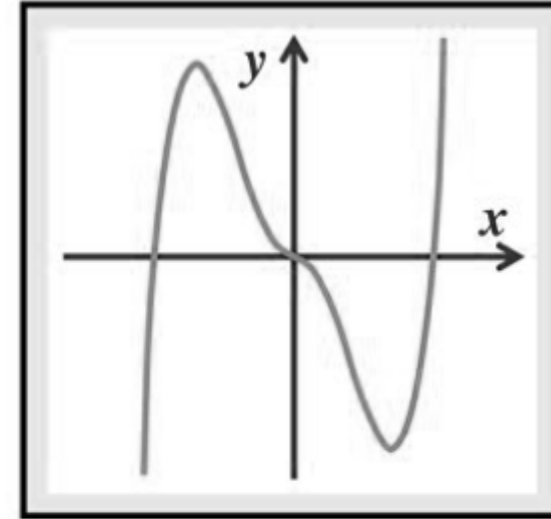
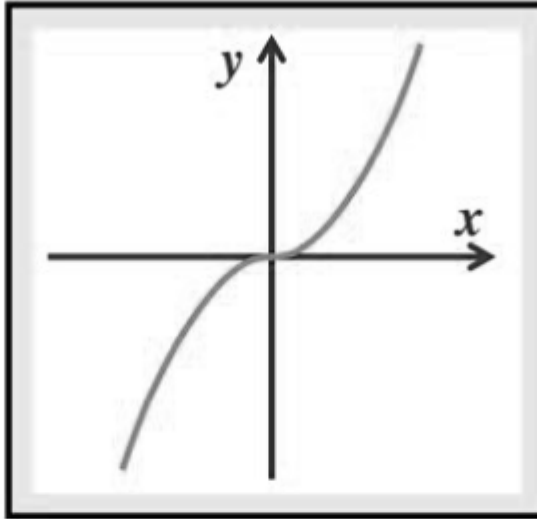
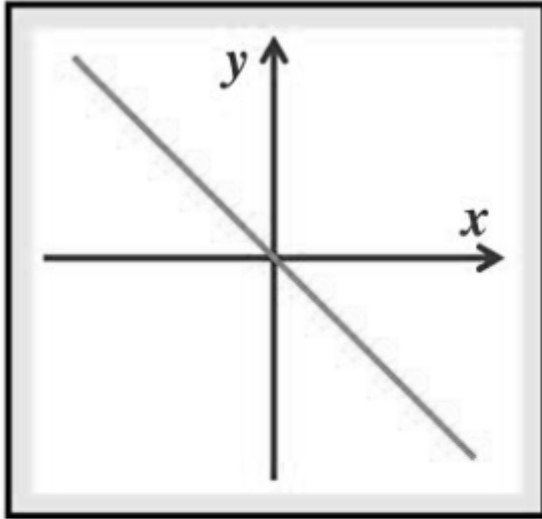
$$= -x^3 - 4x$$

$$= -(x^3 + 4x)$$

$$= -f(x)$$

$f(-x) = -f(x)$, so the function is odd.

Graphs of Odd Functions

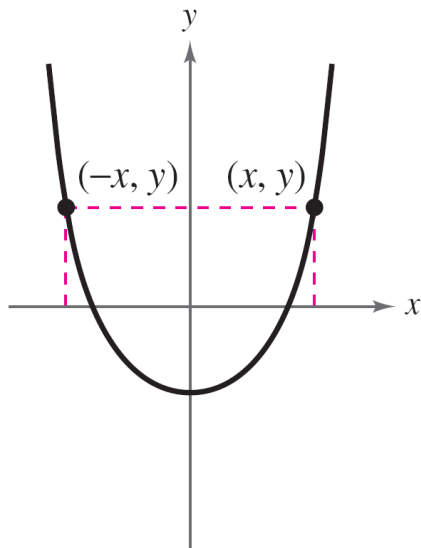


What do you notice about these graphs?

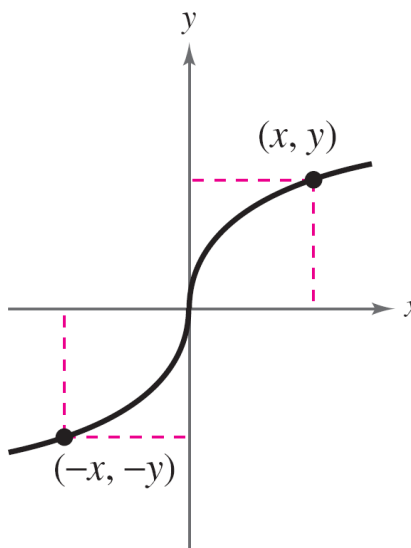
The graphs of all odd functions have order 2 rotational symmetry about the origin.

Important Notes – Even and Odd Functions

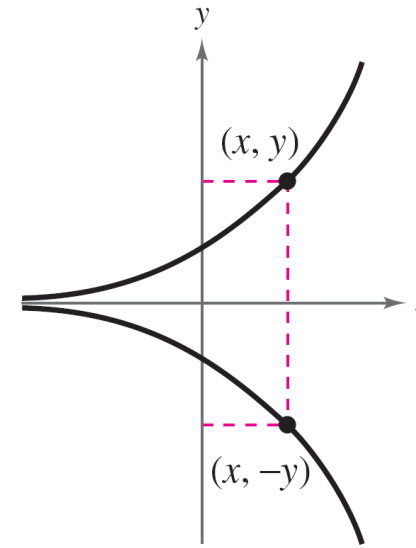
- A graph that is symmetric with respect to the x -axis is not the graph of a function (except for the graph of $y = 0$). These three types of symmetry



Symmetric to y -axis
Even function



Symmetric to origin
Odd function



Symmetric to x -axis
Not a function

Properties of Even and Odd Functions

- ☐ If x has an even power, this means that the function is even
- ☐ If x has an odd power, this means that the function is odd
- ☐ The sum of even functions gives us an even function
- ☐ The sum of odd functions gives us an odd function

Example— Even and Odd Functions

Determine whether each function is even, odd, or neither:

a. $g(x) = x^3 - x$

b. $h(x) = x^2 + 1$

c. $f(x) = x^3 - 1$

Example– Even and Odd Functions

$$a. g(x) = x^3 - x$$

- x^3 is an x with odd power “3”. So, it is odd function.
- x is an x with odd power “1”. So, it is odd function.

So, $x^3 - x$ is odd function because **The sum of odd functions gives us an odd function.**

Example– Even and Odd Functions

b. $h(x) = x^2 + 1$

- x^2 is an x with even power “2”. So, it is even function.
- 1 is a constant number. So, it is even function.

So, $x^2 + 1$ is even function because **The sum of even functions gives us an even function.**

Example– Even and Odd Functions

c. $f(x) = x^3 - 1$

- x^3 is an x with odd power “3”. So, it is odd function.
- 1 is a constant number. So, it is even function.

So, $x^3 - 1$ is NOT EVEN function OR ODD function.

Periodic Functions

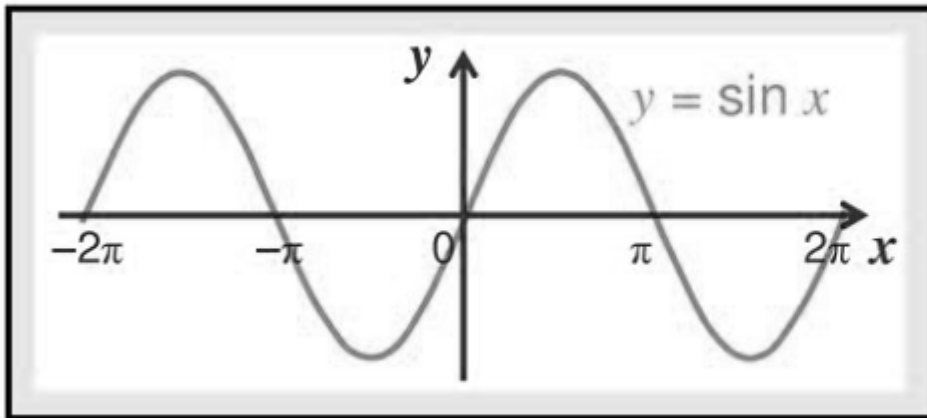
Periodic Functions

A function with a pattern that repeats at regular intervals is called a **periodic function**.

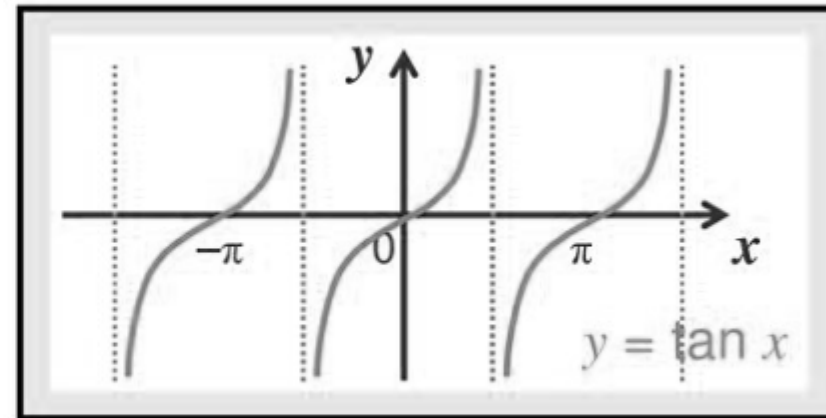
The interval over which a periodic function repeats is called the **period** of the function.

Graphs of Periodic Functions

For example:



This graph has a period of 2π .



This graph has a period of π .

Fourier Series

First let us write down a full series of sines and cosines, with a name for all coefficients:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx\frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx\frac{\pi}{L}\right)$$

Where:

- $f(x)$ is the function we want (such as a square wave)
- L is **half of the period** of the function
- a_0 , a_n and b_n are **coefficients** that we need to calculate!

Coefficients of Fourier Series

To find the coefficients a_0 , a_n and b_n we use these formulas:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(nx \frac{\pi}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(nx \frac{\pi}{L}\right) dx$$

□ If the function $f(x)$ is even,

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(nx\frac{\pi}{L}\right) dx$$

$$b_n = 0$$

□ If the function $f(x)$ is odd,

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(nx \frac{\pi}{L}\right) dx$$

Extra Practice – Example 1

Find the Fourier series for the following function:

$$f(x) = x^2$$

$$-1 < x < 1$$

Note that:

$$\int_0^1 x^2 \cos(n\pi x) dx = \left[\frac{2 \cos(n\pi)}{n^2 \pi^2} \right]$$

Extra Practice – Example 1

Find the Fourier series for the following function:

$$f(x) = x^2$$

$$-1 < x < 1$$

Note that:

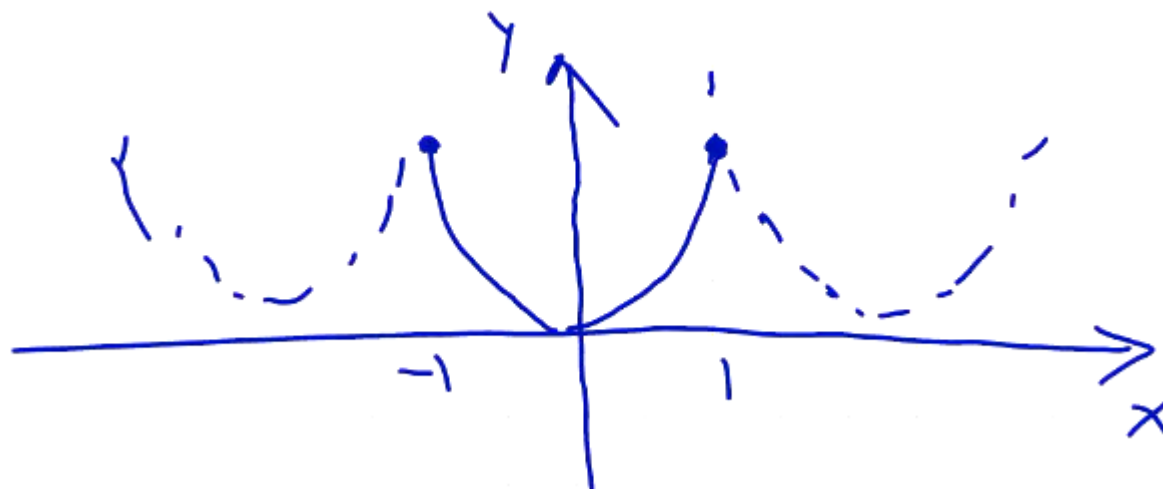
$$\int_0^1 x^2 \cos(n\pi x) dx = \left[\frac{2 \cos(n\pi)}{n^2 \pi^2} \right]$$

Extra Practice – Example 1

$$f(x) = x^2$$

$$2L = 2$$

$$L = 1$$



Extra Practice – Example 1 Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx\frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx\frac{\pi}{L}\right)$$

□ The function $f(x) = x^2$ is even

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(nx\frac{\pi}{L}\right) dx$$

$$b_n = 0$$

Extra Practice – Example 1 Solution

$$\begin{aligned}a_0 &= \frac{2}{L} \int_0^L f(x) dx = \frac{2}{1} \int_0^1 x^2 dx \\&= 2 \left[\frac{x^3}{3} \right]_0^1 \\&= 2 \left[\frac{1}{3} - 0 \right] \\a_0 &= \boxed{\frac{2}{3}}\end{aligned}$$

Extra Practice – Example 1 Solution

$$\begin{aligned}a_0 &= \frac{2}{L} \int_0^L f(x) dx = \frac{2}{1} \int_0^1 x^2 dx \\&= 2 \left[\frac{x^3}{3} \right]_0^1 \\&= 2 \left[\frac{1}{3} - 0 \right] \\a_0 &= \boxed{\frac{2}{3}}\end{aligned}$$

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Extra Practice – Example 1 Solution

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(nx \frac{\pi}{L}\right) dx$$

$$= \frac{2}{1} \int_0^1 x^2 \cos\left(nx \frac{\pi}{1}\right) dx$$

$$= 2 \int_0^1 x^2 \cos(nx\pi) dx$$

$$= 2 \left[\frac{2 \cos(n\pi)}{n^2 \pi^2} \right] = \frac{4}{n^2 \pi^2} (-1)^n$$

Coefficients Values

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{4}{n^2 \pi^2} (-1)^n$$

$$b_n = 0$$

Extra Practice – Example 1 Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx \frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx \frac{\pi}{L}\right)$$

$$= \frac{\frac{2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (-1)^n \cos\left(nx \frac{\pi}{L}\right) + 0$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} (-1)^n \cos(nx \pi)$$

Extra Practice – Example 2

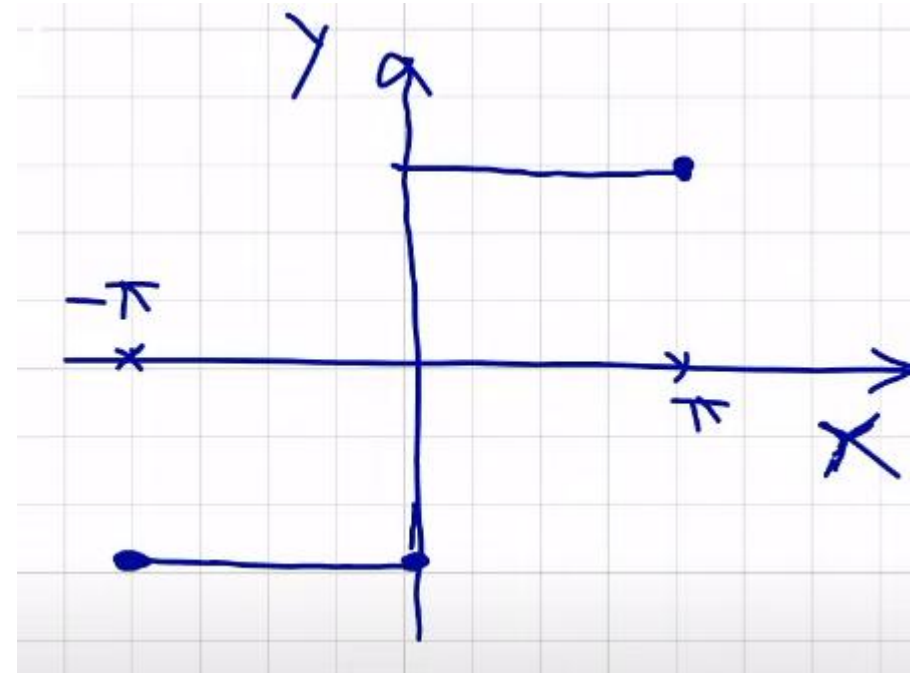
Find the Fourier series for the following function:

$$f(x) = \begin{cases} 2 & 0 \leq x \leq \pi \\ -2 & -\pi \leq x \leq 0 \end{cases}$$

Extra Practice – Example 2 Solution

$$f(x) = \begin{cases} 2 & 0 \leq x \leq \pi \\ -2 & -\pi \leq x \leq 0 \end{cases}$$

$$L = \pi$$



Extra Practice – Example 2 Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(nx\frac{\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(nx\frac{\pi}{L}\right)$$

□ The function $f(x)$ is odd

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(nx\frac{\pi}{L}\right) dx$$

Extra Practice – Example 2 Solution

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(nx \frac{\pi}{L}\right) dx$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 (-2) \sin(nx) dx + \int_0^{\pi} (2) \sin(nx) dx \right]$$

$$b_n = \frac{1}{\pi} \left[\left[2 \frac{\cos(nx)}{n} \right]_{-\pi}^0 + \left[\frac{-2 \cos(nx)}{n} \right]_0^{\pi} \right]$$

Extra Practice – Example 2 Solution

$$b_n = \frac{1}{\pi} \left[\left[\frac{2 \cos(nx)}{n} \right]_{-\pi}^0 + \left[\frac{-2 \cos(nx)}{n} \right]_0^{\pi} \right]$$

$$b_n = \frac{1}{\pi} \left[\left[\frac{2}{n} - \frac{2}{n} (-1)^n \right] + \left[\frac{-2}{n} (-1)^n - \frac{-2}{n} \right] \right]$$

Extra Practice – Example 2 Solution

$$b_n = \frac{1}{\pi} \left[\frac{4}{n} - \frac{4}{n} (-1)^n \right]$$

$$= \frac{4}{\pi n} [1 - (-1)^n]$$

Coefficient Values

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{4}{\pi n} [1 - (-1)^n]$$

Extra Practice – Example 2 Solution

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(nx \frac{\pi}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} [1 - (-1)^n] \sin(nx)$$

Lecture Summary

- Essential information and Important notes for the Fourier series
- Even function and odd function
- Periodic Function
- Fourier series
- Extra Practice



Thank You